Computing Chromatic Polynomials

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for CS 1501, University of Pittsburgh

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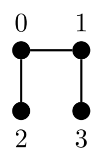
Consider the graph, G, defined as follows:




```
Vertices = 4
Edges = 3
0: 1,2
1: 0,3
2: 0
3: 1
```

In general, we can find the chromatic polynomial by either reducing G to compositions of null graphs or complete graphs. Practically speaking, we will reach a solution faster if we consider the edge density of the graph, and proceed either by reducing to null graphs if our original graph is closer to a null graphs, and complete graphs if it is closer to a omplete graph.

We note that a complete graph (or "clique") with $V = 4$ vertices would have $\frac{(V)(V-1)}{2} = 6$ edges, while a null graph would obviously have 0 edges. Since G has exactly $\frac{(V)(V-1)}{4} = 3$ edges, we can choose to find the chromatic polynomial either reducing to null graphs or complete graphs. We will consider each approach in turn.

1 Reducing to null graphs

Start by selecting a pair of adjacent vertices (u, v) and removing their edge from the original graph. The **Fundamental Reduction Theorem** for edge-removing reductions states that:

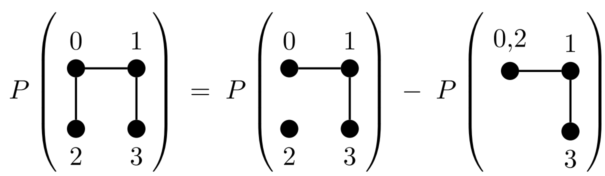
$$P(G, x) = P(G - uv, x) - P(G_{uv}, x)$$

In other words, the chromatic polynomial of the original graph G can be expressed as the **difference** of the chromatic polynomials of two new graphs:

1. $G - uv$, which is constructed from G by **removing** edge (u, v); and

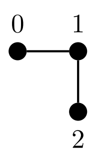
2. G_{uv}, which is constructed from G by merging vertices u and v.

Selecting $(0, 2)$ gives us:



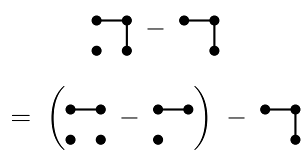
Since not all of these composite graphs are null, we must recurse on these graphs until we have expressed $P(G)$ in terms of a linear combination of the chromatic polynomials of null graphs.

Note that, from an implementation standpoint, in order to represent the second graph as an object of type `Graph`, we would need to logically renumber the vertices, so the internal representation of the third graph above (G_{uv}) would become simply:

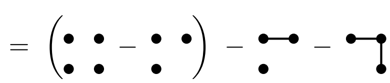



```
Vertices = 3
Edges = 2
0: 1
1: 0,2
2: 1
```

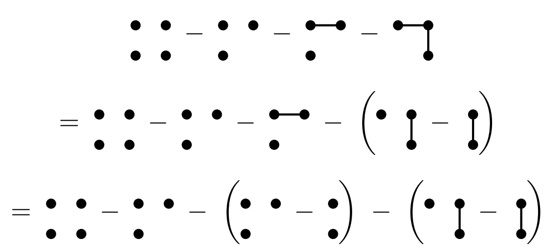
On each recursive call, we further decompose one of our non-null graphs. Fundamentally, order doesn't matter, but let's decompose the more complex graph first. For the purposes of brevity and clarity, we will omit the $P()$ notation moving forward.

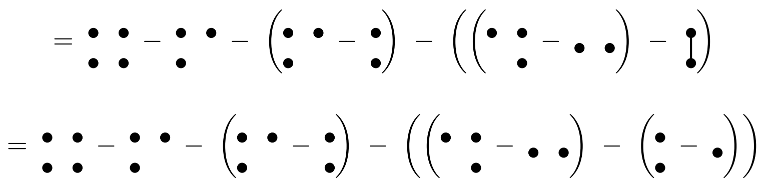


Eventually, we will have reduced this first graph to a null graph:

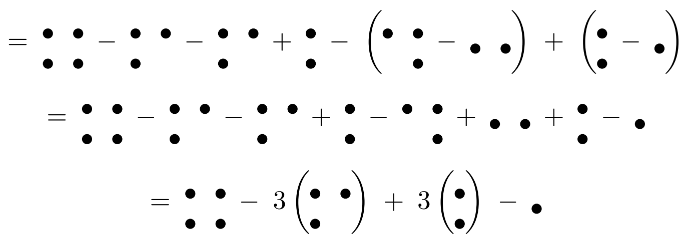


However, we still have some non-null graphs, so we continue recursively reducing them.





Now that we have only null graphs, we simply apply algebraic principles to their chromatic polynomials to simplify this expression:



Lastly, to compute the chromatic polynomial of the original graph G, we plug in the chromatic polynomials of the composite null graphs. Recall that

$$P\left(\text{Null}(n), x\right) \;=\; x^n;$$

that is, the chromatic polynomial of a null graph with n vertices is x^n. So,

$$\begin{aligned} P(G) &= x^4 - 3\left(x^3\right) + 3\left(x^2\right) - x^1 \\ &= x^4 - 3x^3 + 3x^2 - x \end{aligned}$$

2 Reducing to complete graphs

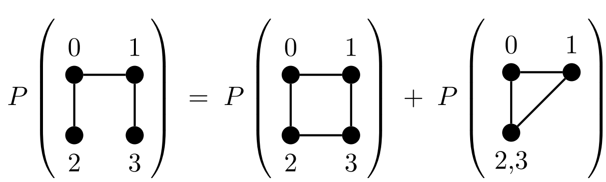
Start by selecting a pair of non-adjacent vertices (u, v) and adding such an edge to the original graph. The **Fundamental Reduction Theorem** for edge-adding reductions states that:

$$P(G, x) = P(G + uv, x) + P(G_{uv}, x)$$

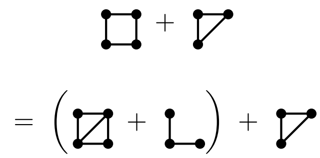
In other words, the chromatic polynomial of the original graph G can be expressed as the **sum** of the chromatic polynomials of two new graphs:

1. $G + uv$, which is constructed from G by **adding** edge (u, v); and
2. G_{uv}, which is constructed from G by merging vertices u and v.

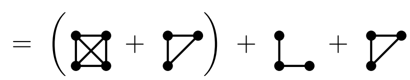
Selecting $(2, 3)$ gives us:



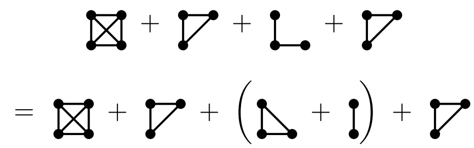
Since not all of these composite graphs are complete, we must recurse on these graphs until we have expressed $P(G)$ in terms of a linear combination of the chromatic polynomials of complete graphs. On each recursive call, we further decompose one of our non-complete graphs.



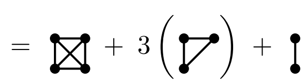
Eventually, we will have reduced this first graph to a complete graph:



However, we still have some non-complete graphs, so we continue recursively reducing them.



Now that we have only complete graphs, we simply apply algebraic principles to their chromatic polynomials to simplify this expression:



Lastly, to compute the chromatic polynomial of the original graph G, we plug in the chromatic polynomials of the composite complete graphs. Recall that

$$P\left(\text{Complete}(n), x\right) \;=\; x(x-1)(x-2)\ldots(x-n+1);$$

that is, the chromatic polynomial of a complete graph with n vertices is $x(x-1)(x-2)\ldots(x-n+1)$. So,

$$\begin{aligned} P(G) &= x(x-1)(x-2)(x-3) + 3\left(x(x-1)(x-2)\right) + x(x-1) \\ &= (x^4 - 6x^3 + 11x^2 - 6x) + 3(x^3 - 3x^2 + 2x) + (x^2 - x) \\ &= x^4 - 6x^3 + 11x^2 - 6x + 3x^3 - 9x^2 + 6x + x^2 - x \\ &= x^4 - 3x^3 + 3x^2 - x \end{aligned}$$

which is, happily, the same answer we obtained with the edge-removing reductions.