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CS1501 Recitation Notes Week 5 October 7, 2011

Lagrange Polynomial

One of the curve-fitting methods you'll have to use for Project 3 is the Lagrange polynomial. Say you have n points $((x_1, y_1), ..., (x_n, y_n))$ that you are trying to fit to some curve. Then you can create n polynomials polynomials $p_1, ..., p_n$, where $p_i(x_j) = 0$ for $i \neq j$ but $p_i(x_i) = y_i$. If you sum them all together, then you get a polynomial that goes through those n points. The polynomial is defined as:

$$p_i(x) = y_i \prod_{j \neq i} \frac{x - x_j}{x_i - x_j}$$

And the Lagrange polynomial then is:

$$L(x) = \sum_i p_i(x)$$

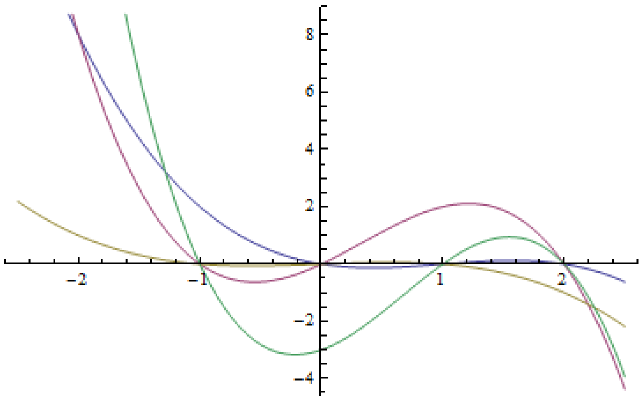
For example, suppose your points are $(-1, 2), (0, -3), (1, 2), (2, -1)$. Then the polynomials are:

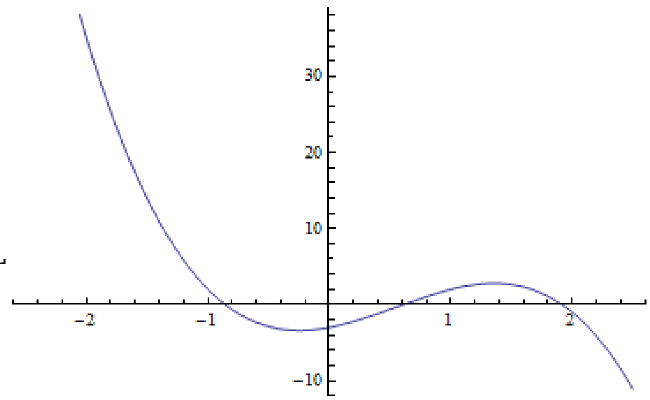
$$\begin{aligned}
p_1(x) &= 2 \cdot \frac{x-1}{-1-1}\frac{x-0}{-1-0}\frac{x-2}{-1-2} = \frac{-1}{3}(x^3 - 3x^2 + 2x) \\
p_2(x) &= -3 \cdot \frac{x+1}{0+1}\frac{x-1}{0-1}\frac{x-2}{0-2} = \frac{-3}{2}(x^3 - 2x^2 - x + 2) \\
p_3(x) &= 2 \cdot \frac{x+1}{1+1}\frac{x-0}{1-0}\frac{x-2}{1-2} = -(x^3 - x^2 - 2x) \\
p_4(x) &= -1 \cdot \frac{x+1}{2+1}\frac{x-0}{2-0}\frac{x-1}{2-1} = \frac{-1}{6}(x^3 - x)
\end{aligned}$$

So the Lagrange polynomial is

$$L(x) = -3x^3 + 5x^2 + 3x - 3$$

And they look like this:





What happens if we add a fifth point, say $(x_5, y_5) = (-2, 0)$? Because $y_5 = 0$, $p_5(x) = 0$, but the other polynomials are multiplied by a factor of $\frac{x+2}{x_i+2}$, causing them all to go through 0 at $x = -2$, ie $p_i(-2) = 0$.

What is the runtime to find the Langrange polynomial? First we have to find all of the polynomials for each of the n points, so there are n of those. To find the coefficient, we have to multiply $y_i \cdot \frac{1}{x_i - x1} \frac{1}{x_i - x_2} \cdot ...$, but since each $x_i - x_j$ is just a constant, this takes time $O(n)$ to multiply all of those together. On the other hand, consider how many multiplies it takes to create the polynomial:

$$\begin{aligned}
&(x - x_1)(x - x_2)(x - x_3) \cdot ... \\
=&(x^2 - (x_1 + x_2)x + x_1x_2)(x - x_3) \cdot ... \\
=&(x^3 - (x_1 + x_2 + x_3)x^2 + (x_1x_2 + x_1x_3 + x_2x_3)x - x_1x_2x_3) \cdot ...
\end{aligned}$$

Notice that the first time we multiply by something with 2 terms (x and constant), the second time we multiply by something with 3 terms (x^2, x, and constant), and the third time we multiply by something with 4 terms (x^3, x^2, x, and constant). So the total number of multiplies increases something like $1+2+3+4... = O(n^2)$. Also notice that once we have the polynomials, we have to add them together, and there are n of them and each has n terms, so the time for this is $O(n^2)$. And so the total time is $O(n^2)$ for the multiplies done n times, plus $O(n^3)$ for the creation of the polynomials n times, plust $O(n^2)$ for the final add, so the total runtime is $O(n^3)$.

Project 3

I don't think there will be another recitation before Project 3 is due, so... You have to implement Lagrange polynomials and two other methods for interpolating between points (cubic spline and least squares line). The project has a graphical part that is very similar to Project 1, so if you have that working you can use that... if not, you might want to try to get that working quickly. For the Langrange polynomial, Dr. Aronis wants you to implement a polynomial class so that you can easily add, multiply, and evaluate polynomials. If done correctly this should make the rest of the project easier.