Functions II

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Functions

- **Definition**: Let A and B be two sets. A function from A to B, denoted \( f : A \rightarrow B \), is an assignment of exactly one element of B to each element of A. We write \( f(a) = b \) to denote the assignment of b to an element a of A by the function f.
Functions

- **Definition**: Let $A$ and $B$ be two sets. A function from $A$ to $B$, denoted $f : A \rightarrow B$, is an assignment of exactly one element of $B$ to each element of $A$. We write $f(a) = b$ to denote the assignment of $b$ to an element $a$ of $A$ by the function $f$.

![Diagram]

Injective function

**Definition**: A function $f$ is said to be **one-to-one**, or **injective**, if and only if $f(x) = f(y)$ implies $x = y$ for all $x, y$ in the domain of $f$. A function is said to be an injection if it is one-to-one.

**Alternative**: A function is one-to-one if and only if $f(x) \neq f(y)$, whenever $x \neq y$. This is the contrapositive of the definition.

![Diagram]
**Surjective function**

**Definition:** A function \( f \) from \( A \) to \( B \) is called **onto**, or **surjective**, if and only if for every \( b \in B \) there is an element \( a \in A \) such that \( f(a) = b \).

**Alternative:** all co-domain elements are covered

![Diagram of surjective function]

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**Bijective functions**

**Definition:** A function \( f \) is called a **bijection** if it is both one-to-one (injection) and onto (surjection).

![Diagram of bijective function]
Bijective functions

Example 1:
• Let $A = \{1,2,3\}$ and $B = \{a,b,c\}$
  – Define $f$ as
    • $1 \rightarrow c$
    • $2 \rightarrow a$
    • $3 \rightarrow b$
• Is $f$ a bijection?
• Yes. It is both one-to-one and onto.
Example 2:
• Define $g : W \rightarrow W$ (whole numbers), where
  $g(n) = \lfloor n/2 \rfloor$ (floor function).
  
  - $0 \rightarrow \lfloor 0/2 \rfloor = \lfloor 0 \rfloor = 0$
  - $1 \rightarrow \lfloor 1/2 \rfloor = \lfloor 1/2 \rfloor = 0$
  - $2 \rightarrow \lfloor 2/2 \rfloor = \lfloor 1 \rfloor = 1$
  - $3 \rightarrow \lfloor 3/2 \rfloor = \lfloor 3/2 \rfloor = 1$
  
  ...

• Is $g$ a bijection?

– No. $g$ is onto but not 1-1 ($g(0) = g(1) = 0$ however $0 \neq 1$.)
Bijective functions

**Theorem:** Let \( f \) be a function \( f: A \rightarrow A \) from a set \( A \) to itself, where \( A \) is finite. Then \( f \) is one-to-one if and only if \( f \) is onto.

**Assume**

\( A \text{ is finite and } f \text{ is one-to-one (injective)} \)

- Is \( f \) an **onto function (surjection)**?

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**Proof:**

\( A \text{ is finite and } f \text{ is one-to-one (injective)} \)

- Is \( f \) an onto function (surjection)?

  - **Yes.** Every element points to exactly one element. Injection assures they are different. So we have \(|A|\) different elements \( A \) points to. Since \( f: A \rightarrow A \) the co-domain is covered thus the function is also a surjection (and a bijection)

\( A \text{ is finite and } f \text{ is an onto function} \)

- Is the function one-to-one?
**Bijective functions**

**Theorem.** Let f be a function f: A \(\rightarrow\) A from a set A to itself, where A is finite. Then f is one-to-one if and only if f is onto.

**Proof:**

\(\rightarrow\) **A is finite and f is one-to-one (injective)**

- Is f an onto function (surjection)?
- **Yes.** Every element points to exactly one element. Injection assures they are different. So we have \(|A|\) different elements A points to. Since f: A \(\rightarrow\) A the co-domain is covered thus the function is also a surjection (and a bijection)

\(\leftarrow\) **A is finite and f is an onto function**

- Is the function one-to-one?
- **Yes.** Every element maps to exactly one element and all elements in A are covered. Thus the mapping must be one-to-one

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**Please note the above is not true when A is an infinite set.**

- **Example:**
  - \(f: \mathbb{Z} \rightarrow \mathbb{Z}\), where \(f(z) = 2 \times z\).
  - f is one-to-one but not onto.
    - \(1 \rightarrow 2\)
    - \(2 \rightarrow 4\)
    - \(3 \rightarrow 6\)
  - 3 has no pre-image.
Functions on real numbers

**Definition**: Let $f_1$ and $f_2$ be functions from $A$ to $\mathbb{R}$ (reals). Then $f_1 + f_2$ and $f_1 \cdot f_2$ are also functions from $A$ to $\mathbb{R}$ defined by

- $(f_1 + f_2)(x) = f_1(x) + f_2(x)$
- $(f_1 \cdot f_2)(x) = f_1(x) \cdot f_2(x)$.

**Examples**:
- Assume
  - $f_1(x) = x - 1$
  - $f_2(x) = x^3 + 1$
  then
    - $(f_1 + f_2)(x) = x^3 + x$
    - $(f_1 \cdot f_2)(x) = x^4 - x^3 + x - 1$.

Increasing and decreasing functions

**Definition**: A function $f$ whose domain and codomain are subsets of real numbers is **strictly increasing** if $f(x) > f(y)$ whenever $x > y$ and $x$ and $y$ are in the domain of $f$. Similarly, $f$ is called **strictly decreasing** if $f(x) < f(y)$ whenever $x > y$ and $x$ and $y$ are in the domain of $f$.

**Example**:
- Let $g : \mathbb{R} \rightarrow \mathbb{R}$, where $g(x) = 2x - 1$. Is it increasing?
Increasing and decreasing functions

**Definition:** A function f whose domain and codomain are subsets of real numbers is **strictly increasing** if f(x) > f(y) whenever x > y and x and y are in the domain of f. Similarly, f is called **strictly decreasing** if f(x) < f(y) whenever x > y and x and y are in the domain of f.

**Example:**
- Let g : \( \mathbb{R} \rightarrow \mathbb{R} \), where \( g(x) = 2x - 1 \). Is it increasing?
- **Proof.**
  - For \( x > y \) holds \( 2x > 2y \) and subsequently \( 2x-1 > 2y-1 \)
  - Thus g is strictly increasing.

**Note:** Strictly increasing and strictly decreasing functions are one-to-one.

**Why?**
Increasing and decreasing functions

**Definition**: A function $f$ whose domain and codomain are subsets of real numbers is **strictly increasing** if $f(x) > f(y)$ whenever $x > y$ and $x$ and $y$ are in the domain of $f$. Similarly, $f$ is called **strictly decreasing** if $f(x) < f(y)$ whenever $x > y$ and $x$ and $y$ are in the domain of $f$.

**Note**: Strictly increasing and strictly decreasing functions are one-to-one.

**Why?**
One-to-one function: A function is one-to-one if and only if $f(x) \neq f(y)$, whenever $x \neq y$.

Identity function

**Definition**: Let $A$ be a set. The **identity function** on $A$ is the function $i_A : A \rightarrow A$ where $i_A(x) = x$.

**Example**:
- Let $A = \{1,2,3\}$

**Then**:
- $i_A(1) = ?$
Identity function

**Definition:** Let $A$ be a set. The *identity function* on $A$ is the function $i_A: A \rightarrow A$ where $i_A(x) = x$.

**Example:**
- Let $A = \{1,2,3\}$
- Then:
  - $i_A(1) = 1$
  - $i_A(2) = 2$
  - $i_A(3) = 3$.

Bijective functions

**Definition:** A function $f$ is called a *bijection* if it is both one-to-one and onto.
Inverse functions

**Definition:** Let \( f \) be a bijection from set \( A \) to set \( B \). The inverse function of \( f \) is the function that assigns to an element \( b \) from \( B \) the unique element \( a \) in \( A \) such that \( f(a) = b \). The inverse function of \( f \) is denoted by \( f^{-1} \). Hence, \( f^{-1}(b) = a \), when \( f(a) = b \). If the inverse function of \( f \) exists, \( f \) is called invertible.

Note: If \( f \) is not a bijection then it is not possible to define the inverse function of \( f \). Why?

Assume \( f \) is not one-to-one:

?
Inverse functions

Note: if \( f \) is not a bijection then it is not possible to define the inverse function of \( f \). **Why?**

**Assume \( f \) is not one-to-one:**
Inverse is not a function. One element of \( B \) is mapped to two different elements.

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Inverse functions

Note: if \( f \) is not a bijection then it is not possible to define the inverse function of \( f \). **Why?**

**Assume \( f \) is not onto:**

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Inverse functions

Note: if \( f \) is not a bijection then it is not possible to define the inverse function of \( f \). Why?

**Assume \( f \) is not onto:**
Inverse is not a function. One element of \( B \) is not assigned any value in \( B \).

**Example 1:**
- Let \( A = \{1,2,3\} \) and \( i_A \) be the identity function

\[
\begin{align*}
  i_A(1) &= 1 & i_A^{-1}(1) &= 1 \\
  i_A(2) &= 2 & i_A^{-1}(2) &= 2 \\
  i_A(3) &= 3 & i_A^{-1}(3) &= 3 
\end{align*}
\]

- Therefore, the inverse function of \( i_A \) is \( i_A \).
Inverse functions

Example 2:
• Let \( g : \mathbb{R} \rightarrow \mathbb{R} \), where \( g(x) = 2x - 1 \).
• What is the inverse function \( g^{-1} \)?

Approach to determine the inverse:
\[
y = 2x - 1 \quad \Rightarrow \quad y + 1 = 2x \\
\quad \Rightarrow \quad \frac{y+1}{2} = x
\]
• Define \( g^{-1}(y) = \frac{y+1}{2} \)

Test the correctness of inverse:
• \( g(3) = \ldots \)
Inverse functions

Example 2:
• Let \( g : \mathbb{R} \rightarrow \mathbb{R} \), where \( g(x) = 2x - 1 \).
• What is the inverse function \( g^{-1} \)?

Approach to determine the inverse:
\[
y = 2x - 1 \implies y + 1 = 2x \\
\implies (y+1)/2 = x
\]
• Define \( g^{-1}(y) = x = (y+1)/2 \)

Test the correctness of inverse:
• \( g(3) = 2*3 - 1 = 5 \)
• \( g^{-1}(5) = (5+1)/2 = 3 \)
• \( g(10) = \)
Inverse functions

Example 2:
• Let $g : \mathbb{R} \rightarrow \mathbb{R}$, where $g(x) = 2x - 1$.
• What is the inverse function $g^{-1}$?

Approach to determine the inverse:

\[
y = 2x - 1 \implies y + 1 = 2x \\
=> (y+1)/2 = x
\]
• Define $g^{-1}(y) = x = (y+1)/2$

Test the correctness of inverse:

• $g(3) = 2*3 - 1 = 5$
• $g^{-1}(5) = (5+1)/2 = 3$
• $g(10) = 2*10 - 1 = 19$
• $g^{-1}(19) = (19+1)/2 = 10$. 

Composition of functions

**Definition:** Let \( f \) be a function from set \( A \) to set \( B \) and let \( g \) be a function from set \( B \) to set \( C \). The *composition of the functions* \( g \) and \( f \), denoted by \( g \circ f \) is defined by

\[
(g \circ f)(a) = g(f(a)).
\]

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**Example 1:**

- Let \( A = \{1,2,3\} \) and \( B = \{a,b,c,d\} \)

\[
g : A \rightarrow A, \quad f : A \rightarrow B
\]

\[
1 \rightarrow 3, \quad 1 \rightarrow b \\
2 \rightarrow 1, \quad 2 \rightarrow a \\
3 \rightarrow 2, \quad 3 \rightarrow d
\]

\[
f \circ g : A \rightarrow B:
\]

- \( 1 \rightarrow \)
Composition of functions

Example 1:
• Let $A = \{1,2,3\}$ and $B = \{a,b,c,d\}$

\[
g : A \rightarrow A, \quad f : A \rightarrow B
\]
\[
1 \rightarrow 3 \quad 1 \rightarrow b
2 \rightarrow 1 \quad 2 \rightarrow a
3 \rightarrow 2 \quad 3 \rightarrow d
\]

$f \circ g : A \rightarrow B$:
• $1 \rightarrow d$
• $2 \rightarrow$
• $3 \rightarrow$
Composition of functions

Example 1:
• Let \( A = \{1,2,3\} \) and \( B = \{a,b,c,d\} \)

\[
\begin{align*}
g & : A \to A, \quad f: A \to B \\
1 & \mapsto 3 \quad 1 \mapsto b \\
2 & \mapsto 1 \quad 2 \mapsto a \\
3 & \mapsto 2 \quad 3 \mapsto d \\
\end{align*}
\]

\( f \circ g : A \to B: \)
• \( 1 \mapsto d \)
• \( 2 \mapsto b \)
• \( 3 \mapsto a \)

Composition of functions

Example 2:
• Let \( f \) and \( g \) be two functions from \( Z \to Z \), where
  • \( f(x) = 2x \) and \( g(x) = x^2 \).
  • \( f \circ g : Z \to Z \)

\[
\begin{align*}
(f \circ g)(x) &= f(g(x)) \\
&= f(x^2) \\
&= 2(x^2)
\end{align*}
\]

• \( g \circ f : Z \to Z \)
• \( (g \circ f)(x) = ? \)
Composition of functions

**Example 2:**
- Let \( f \) and \( g \) be two functions from \( \mathbb{Z} \) to \( \mathbb{Z} \), where
- \( f(x) = 2x \) and \( g(x) = x^2 \).
- \( f \circ g : \mathbb{Z} \rightarrow \mathbb{Z} \)
- \((f \circ g)(x) = f(g(x)) = f(x^2) = 2(x^2)\)
- \( g \circ f : \mathbb{Z} \rightarrow \mathbb{Z} \)
- \((g \circ f)(x) = g(f(x)) = g(2x) = (2x)^2\) Note that the order of the function composition matters

Composition of functions

**Example 3:**
- \((f \circ f^{-1})(x) = x \) and \((f^{-1} \circ f)(x) = x\), for all \( x \).
- Let \( f : \mathbb{R} \rightarrow \mathbb{R} \), where \( f(x) = 2x - 1 \) and \( f^{-1}(x) = \frac{(x+1)}{2} \).
- \((f \circ f^{-1})(x)= f( f^{-1}(x)) = f( \frac{(x+1)}{2} ) = 2( \frac{(x+1)}{2} ) - 1 = (x+1) - 1 = x\)
Composition of functions

Example 3:
- \((f \circ f^{-1})(x) = x\) and \((f^{-1} \circ f)(x) = x\), for all \(x\).

Let \(f : \mathbb{R} \to \mathbb{R}\), where \(f(x) = 2x - 1\) and \(f^{-1}(x) = (x+1)/2\).

- \((f \circ f^{-1})(x) = f(f^{-1}(x))\)
  \[= f\left(\frac{x+1}{2}\right)\]
  \[= 2\left(\frac{x+1}{2}\right) - 1\]
  \[= (x+1) - 1\]
  \[= x\]

- \((f^{-1} \circ f)(x) = f^{-1}(f(x))\)
  \[= f^{-1}(2x - 1)\]
  \[= (2x)/2\]
  \[= x\]

Some functions

Definitions:
- The floor function assigns a real number \(x\) the largest integer that is less than or equal to \(x\). The floor function is denoted by \(\lfloor x \rfloor\).
- The ceiling function assigns to the real number \(x\) the smallest integer that is greater than or equal to \(x\). The ceiling function is denoted by \(\lceil x \rceil\).

Other important functions:
- Factorials: \(n! = n(n-1)\) such that \(1! = 1\)