CS 441 Discrete Mathematics for CS
Lecture 5

Predicate logic

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Negation of quantifiers

English statement:

- Nothing is perfect.
- **Translation:** ¬ ∃x Perfect(x)

Another way to express the same meaning:

- **Everything ...**

Negation of quantifiers

English statement:

- Nothing is perfect.
- **Translation:** $\neg \exists x$ Perfect(x)

Another way to express the same meaning:

- **Everything is imperfect.**
- **Translation:** $\forall x \neg$ Perfect(x)

Conclusion: $\neg \exists x\ P(x)$ is equivalent to $\forall x \neg P(x)$

Negation of quantifiers

English statement:

- It is not the case that all dogs are fleabags.
- **Translation:** $\neg \forall x$ Dog(x) → Fleabag(x)

Another way to express the same meaning:

- There is a dog that …

Negation of quantifiers

English statement:

- It is not the case that all dogs are fleabags.
- **Translation:** ¬ ∀x Dog(x) → Fleabag(x)

Another way to express the same meaning:

- There is a dog that is not a fleabag.
- **Translation:** ∃x Dog(x) ∧ ¬ Fleabag(x)

- Logically equivalent to:
  - ∃x ¬ (Dog(x) → Fleabag(x))

Conclusion: ¬ ∀x P (x) is equivalent to ∃x ¬ P(x)

Negation of quantified statements (aka DeMorgan Laws for quantifiers)

Negation	Equivalent
¬∃x P(x)	∀x ¬P(x)
¬∀x P(x)	∃x ¬P(x)

Formal and informal proofs

Theorems and proofs

- The truth value of some statement about the world is obvious and easy to assign
- The truth of other statements may not be obvious, …

  …. But it may still follow (be derived) from known facts about the world

To show the truth value of such a statement following from other statements we need to provide **a correct supporting argument**

- **a proof**

Important questions:

- When is the argument correct?
- How to construct a correct argument, what method to use?

Theorems and proofs

- **Theorem:** a statement that can be shown to be true.
  - **Typically the theorem looks like this:**

$$\underbrace{(p1 \wedge p2 \wedge p3 \wedge \dots \wedge pn)}_{\textbf{Premises}} \rightarrow \underbrace{q}_{\textbf{conclusion}}$$

- **Example:**

  Fermat's Little theorem:
  - If p is a prime and a is an integer not divisible by p, then: $a^{p-1} \equiv 1 \mod p$

Theorems and proofs

- **Theorem:** a statement that can be shown to be true.
  - **Typically the theorem looks like this:**

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- **Example:**

  Premises (hypotheses)

  Fermat's Little theorem:
  - If p is a prime and a is an integer not divisible by p, then: $\boxed{a^{p-1} \equiv 1 \mod p}$

  conclusion

Formal proofs

Proof:

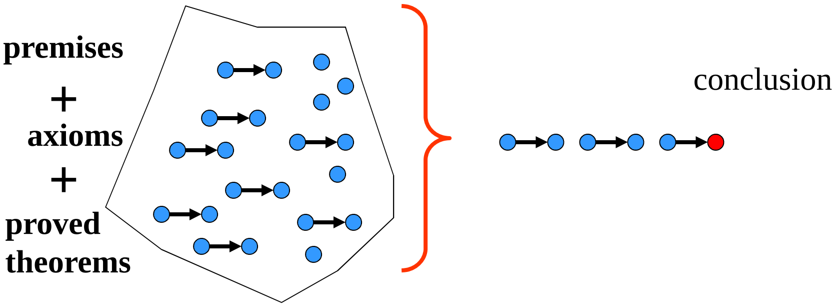
- Provides an argument supporting the validity of the statement
- Proof of the theorem:
  - shows that the conclusion follows from premises
  - May use:
    - **Premises**
    - **Axioms**
    - **Results of other theorems**

Formal proofs:

- steps of the proofs **follow logically** from the set of premises and axioms

Formal proofs

- **Formal proofs:**
  - show that **steps of the proofs follow logically** from the set of hypotheses and axioms



premises

+

axioms

+

proved theorems

conclusion

In this class we assume formal proofs in the propositional logic

Using logical equivalence rules

Proofs based on logical equivalences. A proposition or its part can be transformed using a sequence of equivalence rewrites till some conclusion can be reached.

Example: Show that $(p \wedge q) \rightarrow p$ is a tautology.

- Proof: (we must show $(p \wedge q) \rightarrow p$ <=> T)

$(p \wedge q) \rightarrow p$ <=> $\neg(p \wedge q) \vee p$ Useful

<=> $[\neg p \vee \neg q] \vee p$ DeMorgan

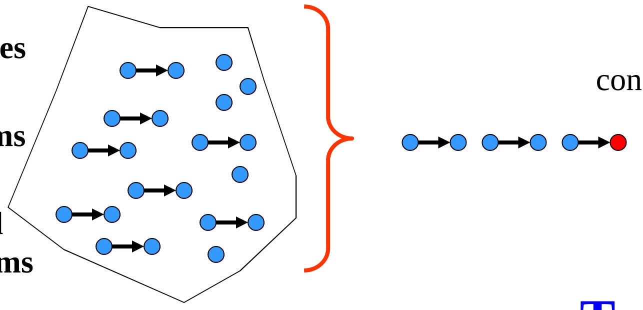
<=> $[\neg q \vee \neg p] \vee p$ Commutative

<=> $\neg q \vee [\neg p \vee p]$ Associative

<=> $\neg q \vee [T]$ Useful

Formal proofs

Allow us to infer new True statements from known True statements



premises

+

axioms

+

proved theorems

conclusion

True ?

True

Rules of inference

Rules of inference:

- **Allow us to infer new True statements from existing True statements**
- **Represent logically valid inference patterns**

Example:

- **Modus Ponens**, or the Law of Detachment
- Rule of inference

$$\begin{array}{l} p \\ \underline{p \to q} \\ \therefore q \end{array}$$

- Given p is true and the implication p → q is true then q is true.

Rules of inference

Rules of inference: logically valid inference patterns

Example;

- **Modus Ponens**, or the Law of Detachment
- Rule of inference

$$\begin{array}{l} p \\ \underline{p \to q} \\ \therefore q \end{array}$$

- Given p is true and the implication p → q is true then q is true.

p	q	p → q
False	*False*	*True*
False	*True*	*True*
True	*False*	*False*
True	*True*	*True*

Rules of inference

Rules of inference: logically valid inference patterns

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Rules of inference

Rules of inference: logically valid inference patterns

Example:

- **Modus Ponens**, or the Law of Detachment
- Rules of inference

$$\begin{array}{l} p \\ \underline{p \rightarrow q} \\ \therefore q \end{array}$$

- Given p is true and the implication p → q is true then q is true.
- **Tautology Form:** $(p \wedge (p \rightarrow q)) \rightarrow q$

Rules of inference

- **Addition**

  $p \rightarrow (p \vee q)$

  $\underline{p\qquad}$
  $\therefore\ p \vee q$

- **Example:** It is below freezing now. Therefore, it is below freezing or raining snow.

- **Simplification**

  $(p \wedge q) \rightarrow p$

  $\underline{\ p \wedge q}$
  $\therefore\ p$

- **Example:** It is below freezing and snowing. Therefore it is below freezing.

Rules of inference

- **Modus Tollens (modus ponens for the contrapositive)**

  $[\ \neg q \wedge (p \rightarrow q)] \rightarrow \neg p$

  $\neg q$
  $\underline{\ p \rightarrow q\ }$
  $\therefore\ \neg p$

- **Hypothetical Syllogism**

  $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$

  $p \rightarrow q$
  $\underline{\ q \rightarrow r}$
  $\therefore\ p \rightarrow r$

- **Disjunctive Syllogism**

  $[(p \vee q) \wedge \neg p] \rightarrow q$

  $p \vee q$
  $\underline{\ \neg p\qquad}$
  $\therefore\ q$

Rules of inference

- **Logical equivalences (discussed earlier)**

  A <=> B

  A → B is a tautology

Example: De Morgan Law

$\neg(p \vee q) \iff \neg p \wedge \neg q$

$\neg(p \vee q) \rightarrow \neg p \wedge \neg q$ **is a tautology**

Rules of inference

- A **valid argument** is one built using the rules of inference from premises (hypotheses). When all premises are true the argument should lead us to the correct conclusion.
- $(p1 \wedge p2 \wedge p3 \wedge \ldots \wedge pn) \rightarrow q$
- **How to use the rules of inference?**

Applying rules of inference

Assume the following statements (hypotheses):

- It is not sunny this afternoon and it is colder than yesterday.
- We will go swimming only if it is sunny.
- If we do not go swimming then we will take a canoe trip.
- If we take a canoe trip, then we will be home by sunset.

Show that all these lead to a conclusion:

- We will be home by sunset.

Applying rules of inference

Text:

(1) It is not sunny this afternoon and it is colder than yesterday.
(2) We will go swimming only if it is sunny.
(3) If we do not go swimming then we will take a canoe trip.
(4) If we take a canoe trip, then we will be home by sunset.

Propositions:

- p = It is sunny this afternoon, q = it is colder than yesterday,
  r = We will go swimming , s= we will take a canoe trip
- t= We will be home by sunset

Translation:

- **Assumptions:** (1) $\neg p \wedge q$, (2) ?
- **We want to show:** t

Applying rules of inference

- **Approach:**
- p = It is sunny this afternoon, q = it is colder than yesterday, r = We will go swimming , s= we will take a canoe trip
- t= We will be home by sunset

Translation: “We will go swimming only if it is sunny”.

- Ambiguity: $r \rightarrow p$ or $p \rightarrow r$?
- Sunny is a must before we go swimming
- Thus, if we indeed go swimming it must be sunny, therefore $r \rightarrow p$

Applying rules of inference

Text:

(1) It is not sunny this afternoon and it is colder than yesterday.

(2) We will go swimming only if it is sunny.

(3) If we do not go swimming then we will take a canoe trip.

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Propositions:

- p = It is sunny this afternoon, q = it is colder than yesterday, r = We will go swimming , s= we will take a canoe trip
- t= We will be home by sunset

Translation:

- **Assumptions:** (1) $\neg p \wedge q$, (2) $r \rightarrow p$, (3) $\neg r \rightarrow s$, (4) $s \rightarrow t$
- **We want to show:** t

Proofs using rules of inference

Translations:

- **Assumptions:** $\neg p \wedge q$, $r \rightarrow p$, $\neg r \rightarrow s$, $s \rightarrow t$
- **We want to show:** t

Proof:

- 1. $\neg p \wedge q$ Hypothesis
- 2. $\neg p$ Simplification
- 3. $r \rightarrow p$ Hypothesis
- 4. $\neg r$ Modus tollens (step 2 and 3)
- 5. $\neg r \rightarrow s$ Hypothesis
- 6. s Modus ponens (steps 4 and 5)
- 7. $s \rightarrow t$ Hypothesis
- 8. t Modus ponens (steps 6 and 7)
- **end of proof**

Informal proofs

Proving theorems in practice:

- The steps of the proofs are not expressed in any formal language as e.g. propositional logic
- Steps are argued less formally using English, mathematical formulas and so on
- One must always watch the consistency of the argument made, logic and its rules can often help us to decide the soundness of the argument if it is in question

- **We use (informal) proofs to illustrate different methods of proving theorems**

Methods of proving theorems

Basic methods to prove the theorems:

- **Direct proof**
  - $p \rightarrow q$ is proved by showing that if p is true then q follows
- **Indirect proof**
  - Show the contrapositive $\neg q \rightarrow \neg p$. If $\neg q$ holds then $\neg p$ follows
- **Proof by contradiction**
  - Show that $(p \wedge \neg q)$ contradicts the assumptions
- **Proof by cases**
- **Proofs of equivalence**
  - $p \leftrightarrow q$ is replaced with $(p \rightarrow q) \wedge (q \rightarrow p)$

Sometimes one method of proof does not go through as nicely as the other method. Hint: try more than one approach.

Direct proof

- $p \rightarrow q$ is proved by showing that if p is true then q follows
- **Example:** Prove that "If n is odd, then n^2 is odd."

Proof:

- Assume the hypothesis is true, i.e. suppose n is odd.
- Then $n = 2k + 1$, where k is an integer.

$$\begin{aligned} n^2 &= (2k+1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1 \end{aligned}$$

- Therefore, n^2 is odd. □

Indirect proof

- To show $p \rightarrow q$ prove its contrapositive $\neg q \rightarrow \neg p$
- Why? **$p \rightarrow q$ and $\neg q \rightarrow \neg p$ are equivalent !!!**
- Assume $\neg q$ is true, show that $\neg p$ is true.

Example: Prove If **$3n + 2$ is odd** then n is odd.

Proof:

- Assume n is even, that is $n = 2k$, where k is an integer.
- Then: $3n + 2 = 3(2k) + 2$
  $= 6k + 2$
  $= 2(3k+1)$
- Therefore **$3n + 2$ is even.**

Indirect proof

- To show $p \rightarrow q$ prove its contrapositive $\neg q \rightarrow \neg p$
- Why? **$p \rightarrow q$ and $\neg q \rightarrow \neg p$ are equivalent !!!**
- Assume $\neg q$ is true, show that $\neg p$ is true.

Example: Prove If $3n + 2$ is odd then n is odd.

Proof:

- Assume n is even, that is $n = 2k$, where k is an integer.
- Then: $3n + 2 = 3(2k) + 2$
  $= 6k + 2$
  $= 2(3k+1)$
- Therefore $3n + 2$ is even.
- We proved **¬ “n is odd” → ¬ “3n + 2 is odd”.** This is equivalent to **“3n + 2 is odd” → “n is odd”.** □