Predicate logic

Propositional logic: review

• **Propositional logic**: a formal language for making logical inferences
• A **proposition** is a statement that is either true or false.
• A **compound proposition** can be created from other propositions using logical connectives
• **The truth of a compound proposition** is defined by truth values of elementary propositions and the meaning of connectives.
• **The truth table for a compound proposition**: table with entries (rows) for all possible combinations of truth values of elementary propositions.
## Tautology and Contradiction

### What is a tautology?
- A compound proposition that is always true for all possible truth values of the propositions is called a **tautology**.

**Example:** \( p \lor \neg p \) is a **tautology**.

<table>
<thead>
<tr>
<th>( p )</th>
<th>( \neg p )</th>
<th>( p \lor \neg p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>

### What is a contradiction?
- A compound proposition that is always false is called a **contradiction**.

**Example:** \( p \land \neg p \) is a **contradiction**.

<table>
<thead>
<tr>
<th>( p )</th>
<th>( \neg p )</th>
<th>( p \land \neg p )</th>
</tr>
</thead>
<tbody>
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<td>T</td>
<td>F</td>
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<td>F</td>
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</table>
Equivalence

• How do we determine that two propositions are equivalent? Their truth values in the truth table are the same.
• Example: \( p \rightarrow q \) is equivalent to \( \neg q \rightarrow \neg p \) (contrapositive)

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>p ( \rightarrow ) q</th>
<th>( \neg q \rightarrow \neg p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
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<tr>
<td>T</td>
<td>F</td>
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<td>T</td>
<td>T</td>
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</table>

• Equivalent statements are important for logical reasoning since they can be substituted and can help us to make a logical argument.

Logical equivalence

• Definition: The propositions p and q are called logically equivalent if \( p \leftrightarrow q \) is a tautology (alternately, if they have the same truth table). The notation \( p \leftrightarrow q \) denotes p and q are logically equivalent.

<table>
<thead>
<tr>
<th>a</th>
<th>b</th>
<th>a ( \rightarrow ) b</th>
<th>( \neg a \rightarrow \neg b )</th>
<th>( (a \rightarrow b) \leftrightarrow (\neg a \rightarrow \neg b) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
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<td>T</td>
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</tbody>
</table>
Important logical equivalences

• **Identity**
  - \( p \land T \iff p \)
  - \( p \lor F \iff p \)

• **Domination**
  - \( p \lor T \iff T \)
  - \( p \land F \iff F \)

• **Idempotent**
  - \( p \lor p \iff p \)
  - \( p \land p \iff p \)

• **Double negation**
  - \( \neg(\neg p) \iff p \)

• **Commutative**
  - \( p \lor q \iff q \lor p \)
  - \( p \land q \iff q \land p \)

• **Associative**
  - \( (p \lor q) \lor r \iff p \lor (q \lor r) \)
  - \( (p \land q) \land r \iff p \land (q \land r) \)
Important logical equivalences

• **Distributive**
  
  - \( p \lor (q \land r) \iff (p \lor q) \land (p \lor r) \)
  
  - \( p \land (q \lor r) \iff (p \land q) \lor (p \land r) \)

• **De Morgan**
  
  - \( \neg (p \lor q) \iff \neg p \land \neg q \)
  
  - \( (p \land q) \iff \neg p \lor \neg q \)

• **Other useful equivalences**
  
  - \( p \lor \neg p \iff T \)
  
  - \( p \land \neg p \iff F \)
  
  - \( p \rightarrow q \iff (\neg p \lor q) \)

Showing logical equivalence

**Example:** Show \((p \land q) \rightarrow p\) is a tautology
In other words \( ((p \land q) \rightarrow p \iff T) \)

**Proof via truth table:**

<table>
<thead>
<tr>
<th>(p)</th>
<th>(q)</th>
<th>(p \land q)</th>
<th>((p \land q) \rightarrow p)</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
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</table>
Showing logical equivalences

- **Equivalences can be used in proofs.** A proposition or its part can be transformed using equivalences and some conclusion can be reached.

**Example:** Show that \((p \land q) \rightarrow p\) is a tautology.
- Proof: (we must show \((p \land q) \rightarrow p \iff T\))

\[
(p \land q) \rightarrow p \iff \neg(p \land q) \lor p
\]
- Useful

\[
\iff [\neg p \lor \neg q] \lor p
\]
- DeMorgan

\[
\iff [\neg q \lor \neg p] \lor p
\]
- Commutative

\[
\iff \neg q \lor [\neg p \lor p]
\]
- Associative

\[
\iff \neg q \lor [T]
\]
- Useful

\[
\iff T
\]
- Domination

Propositional logic

- **Definition:**
  - A **proposition** is a statement that is either true or false.

- **Examples:**
  - Pitt is located in the Oakland section of Pittsburgh.
  - \(5 + 2 = 8\).
  - It is raining today
  - \(2\) is a prime number
  - If (you do not drive over 65 mph) then (you will not get a speeding ticket).

- **Not a proposition:**
  - How are you?
  - \(x + 5 = 3\)
Limitations of the propositional logic

Propositional logic: the world is described in terms of elementary propositions and their logical combinations

Elementary statements:

• Typically refer to objects, their properties and relations. But these are not explicitly represented in the propositional logic

  – Example:
    • “John is a UPitt student.”

    ![Diagram showing John is a UPitt student]

    • Objects and properties are hidden in the statement, it is not possible to reason about them

(1) Statements that must be repeated for many objects

  – In propositional logic these must be exhaustively enumerated

  • Example:
    – If John is a CS UPitt graduate then John has passed cs441

    Translation:
    – John is a CS UPitt graduate \( \Rightarrow \) John has passed cs441

    Similar statements can be written for other Upitt graduates:
    – Ann is a CS Upitt graduate \( \Rightarrow \) Ann has passed cs441
    – Ken is a CS Upitt graduate \( \Rightarrow \) Ken has passed cs441
    – ...

    • What is a more natural solution to express the above knowledge?
Limitations of the propositional logic

(1) Statements that must be repeated for many objects

• Example:
  – If John is a CS UPitt graduate then John has passed cs441

Translation:
  – John is a CS UPitt graduate \(\Rightarrow\) John has passed cs441

Similar statements can be written for other Upitt graduates:
  – Ann is a CS Upitt graduate \(\Rightarrow\) Ann has passed cs441
  – Ken is a CS Upitt graduate \(\Rightarrow\) Ken has passed cs441
  – …

• Solution: make statements with variables
  – If \(x\) is a CS Upitt graduate then \(x\) has passed cs441
  – \(x\) is a CS UPitt graduate \(\Rightarrow\) \(x\) has passed cs441

Limitations of the propositional logic

(2) Statements that define the property of the group of objects

• Example:
  – All new cars must be registered.
  – Some of the CS graduates graduate with honors.

• Solution: make statements with quantifiers
  – Universal quantifier – the property is satisfied by all members of the group
  – Existential quantifier – at least one member of the group satisfy the property
Predicate logic

Remedies the limitations of the propositional logic
• Explicitly models objects and their properties
• Allows to make statements with variables and quantify them

Basic building blocks of the predicate logic:
• Constant – models a specific object
  Examples: “John”, “France”, “7”
• Variable – represents object of specific type (defined by the universe of discourse)
  Examples: x, y
  (universe of discourse can be people, students, numbers)
• Predicate - over one, two or many variables or constants.
  – Represents properties or relations among objects
  Examples: Red(car23), student(x), married(John,Ann)

Predicates

Predicates represent properties or relations among objects

A predicate P(x) assigns a value true or false to each x depending on whether the property holds or not for x.
• The assignment is best viewed as a big table with the variable x substituted for objects from the universe of discourse

Example:
• Assume Student(x) where the universe of discourse are people
  • Student(John) …. T (if John is a student)
  • Student(Ann) …. T (if Ann is a student)
  • Student(Jane) ….. F (if Jane is not a student)
  • ...

Predicates

Assume a predicate $P(x)$ that represents the statement:

• $x$ is a prime number

What are the truth values of:

• $P(2)$ $\ T$
• $P(3)$ $\ T$
• $P(4)$ $\ F$
• $P(5)$ $\ T$
• $P(6)$ $\ F$
• $P(7)$ $\ T$

All statements $P(2), P(3), P(4), P(5), P(6), P(7)$ are propositions

Predicates

Assume a predicate $P(x)$ that represents the statement:

• $x$ is a prime number

What are the truth values of:

• $P(2)$ $\ T$
• $P(3)$ $\ T$
• $P(4)$ $\ F$
• $P(5)$ $\ T$
• $P(6)$ $\ F$
• $P(7)$ $\ T$

Is $P(x)$ a proposition? No. Many possible substitutions are possible.
Predicates

• Predicates can have more arguments which represent the relations between objects

Example:
• Older(John, Peter) denotes ‘John is older than Peter’
  – this is a proposition because it is either true or false
• Older(x,y) – ‘x is older than y’
  – not a proposition, but after the substitution it becomes one

Predicates

• Predicates can have more arguments which represent the relations between objects

Example:
• Let Q(x,y) denote ‘x+5 > y’
  – Is Q(x,y) a proposition?
Predicates

- Predicates can have **more arguments** which represent the **relations between objects**

**Example:**
- Let \( Q(x,y) \) denote ‘\( x + 5 > y \)’
  - Is \( Q(x,y) \) a proposition? **No!**
  - Is \( Q(3,7) \) a proposition? **Yes.** It is true.
  - What is the truth value of:
    - \( Q(3,7) \) \( T \)
    - \( Q(1,6) \) \( F \)
    - \( Q(2,2) \) \( T \)
  - Is \( Q(3,y) \) a proposition? **No!** We cannot say if it is true or false.

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Compound statements in predicate logic

**Compound statements are obtained via logical connectives**

**Examples:**
- \( \text{Student}(\text{Ann}) \land \text{Student}(\text{Jane}) \)
  - **Translation:** “Both Ann and Jane are students”
  - **Proposition:** yes.
- \( \text{Country}(\text{Sienna}) \lor \text{River}(\text{Sienna}) \)
  - **Translation:** “Sienna is a country or a river”
  - **Proposition:** yes.
- \( \text{CS-major}(x) \rightarrow \text{Student}(x) \)
  - **Translation:** “if \( x \) is a CS-major then \( x \) is a student”
  - **Proposition:** no.
Predicates

Important:
• statement P(x) is not a proposition since there are more objects it can be applied to
This is the same as in propositional logic …

… But the difference is:
• predicate logic allows us to explicitly manipulate and substitute for the objects
• Predicate logic permits quantified sentences where variables are substituted for statements about the group of objects

Quantified statements

Predicate logic lets us to make statements about groups of objects
• To do this we use special quantified expressions

Two types of quantified statements:
• universal
  Example: ‘all CS Upitt graduates have to pass cs441’
  – the statement is true for all graduates
• existential
  Example: ‘Some CS Upitt students graduate with honor.’
  – the statement is true for some people
Universal quantifier

**Defn:** The universal quantification of \( P(x) \) is the proposition:
"\( P(x) \) is true for all values of \( x \) in the domain of discourse." The notation \( \forall x \ P(x) \) denotes the universal quantification of \( P(x) \), and is expressed as for every \( x \), \( P(x) \).

**Example:**
- Let \( P(x) \) denote \( x > x - 1 \).
- What is the truth value of \( \forall x \ P(x) \)?
- Assume the universe of discourse of \( x \) is all real numbers.
- **Answer:** Since every number \( x \) is greater than itself minus 1. Therefore, \( \forall x \ P(x) \) is true.

**Quantification converts** a propositional function into a **proposition** by binding a variable to a set of values from the universe of discourse.

**Example:**
- Let \( P(x) \) denote \( x > x - 1 \).
- Is \( P(x) \) a proposition? **No.** Many possible substitutions.
- Is \( \forall x \ P(x) \) a proposition? **Yes.** True if for all \( x \) from the universe of discourse \( P(x) \) is true.
Universally quantified statements

Predicate logic lets us make statements about groups of objects

**Universally quantified statement**

- CS-major(x) → Student(x)
  - **Translation:** “if x is a CS-major then x is a student”
  - **Proposition:** no.

- ∀x CS-major(x) → Student(x)
  - **Translation:** “(For all people it holds that) if a person is a CS-major then she is a student.”
  - **Proposition:** yes.

Existential quantifier

**Definition:** The *existential quantification* of P(x) is the proposition "There exists an element in the domain (universe) of discourse such that P(x) is true." The notation ∃x P(x) denotes the existential quantification of P(x), and is expressed as there is an x such that P(x) is true.

**Example 1:**

- Let T(x) denote x > 5 and x is from Real numbers.
- What is the truth value of ∃x T(x)?
- **Answer:**
  - Since 10 > 5 is true. Therefore, it is true that ∃x T(x).
**Existential quantifier**

**Definition:** The **existential quantification** of $P(x)$ is the proposition "There exists an element in the domain (universe) of discourse such that $P(x)$ is true." The notation $\exists x P(x)$ denotes the existential quantification of $P(x)$, and is expressed as **there is an $x$ such that $P(x)$ is true.**

**Example 2:**
- Let $Q(x)$ denote $x = x + 2$ where $x$ is real numbers
- What is the truth value of $\exists x Q(x)$?
- **Answer:** Since no real number is 2 larger than itself, the truth value of $\exists x Q(x)$ is false.

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**Existentially quantified statements**

**Statements about groups of objects**

**Example:**
- $\text{CS-Upitt-graduate (x)} \land \text{Honor-student(x)}$
  - **Translation:** “$x$ is a CS-Upitt-graduate and $x$ is an honor student”
  - **Proposition:** ?
Quantified statements

Statements about groups of objects

Example:
- CS-Upitt-graduate (x) ∧ Honor-student(x)
  - Translation: “x is a CS-Upitt-graduate and x is an honor student”
  - Proposition: no.

- ∃ x CS-Upitt-graduate (x) ∧ Honor-student(x)
  - Translation: “There is a person who is a CS-Upitt-graduate and who is also an honor student.”
  - Proposition: yes.
### Summary of quantified statements

- When \( \forall x P(x) \) and \( \exists x P(x) \) are true and false?

<table>
<thead>
<tr>
<th>Statement</th>
<th>When true?</th>
<th>When false?</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \forall x P(x) )</td>
<td>( P(x) ) true for all ( x )</td>
<td>There is an ( x ) where ( P(x) ) is false.</td>
</tr>
<tr>
<td>( \exists x P(x) )</td>
<td>There is some ( x ) for which ( P(x) ) is true.</td>
<td>( P(x) ) is false for all ( x ).</td>
</tr>
</tbody>
</table>

Suppose the elements in the universe of discourse can be enumerated as \( x_1, x_2, ..., x_N \) then:
- \( \forall x P(x) \) is true whenever \( P(x_1) \land P(x_2) \land ... \land P(x_N) \) is true
- \( \exists x P(x) \) is true whenever \( P(x_1) \lor P(x_2) \lor ... \lor P(x_N) \) is true.