Cartesian product (review)

Let $A=\{a_1, a_2, \ldots, a_k\}$ and $B=\{b_1, b_2, \ldots, b_m\}$.

The Cartesian product $A \times B$ is defined by a set of pairs

$\{(a_1, b_1), (a_1, b_2), \ldots, (a_1, b_m), \ldots, (a_k, b_m)\}$.

Cartesian product defines a product set, or a set of all ordered arrangements of elements in sets in the Cartesian product.
**Binary relation**

**Definition:** Let A and B be two sets. A **binary relation from A to B** is a subset of a Cartesian product A x B.

- Let R ⊆ A x B means R is a set of ordered pairs of the form (a,b) where a ∈ A and b ∈ B.
- We use the notation **a R b** to denote (a,b) ∈ R and **a ̸R b** to denote (a,b) ∉ R. If a R b, we say a is related to b by R.

**Example:** Let A={a,b,c} and B={1,2,3}.
- Is R={(a,1),(b,2),(c,2)} a relation from A to B? **Yes.**
- Is Q={(1,a),(2,b)} a relation from A to B? **No.**
- Is P={(a,a),(b,c),(b,a)} a relation from A to A? **Yes**

---

**Representing binary relations**

- We can graphically represent a binary relation R as follows:
  - If **a R b** then draw an arrow from a to b.
    
    \[ a \rightarrow b \]

**Example:**
- Let A = \{0, 1, 2\}, B = \{u,v\} and R = \{(0,u), (0,v), (1,v), (2,u)\}
- Note: R ⊆ A x B.
- Graph:

```
   2
  ↘  u
  0   ↘
    v
  1
```
**Representing binary relations**

- We can represent a binary relation $R$ by a **table** showing (marking) the ordered pairs of $R$.

**Example:**
- Let $A = \{0, 1, 2\}$, $B = \{u, v\}$ and $R = \{(0,u), (0,v), (1,v), (2,u)\}$
- **Table:**

<table>
<thead>
<tr>
<th>R</th>
<th>u</th>
<th>v</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>x</td>
<td>x</td>
</tr>
<tr>
<td>1</td>
<td>x</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>x</td>
<td></td>
</tr>
</tbody>
</table>

**or**

<table>
<thead>
<tr>
<th>R</th>
<th>u</th>
<th>v</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
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<tr>
<td>1</td>
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<td>1</td>
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<tr>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

**Relations and functions**

- Relations represent **one to many relationships** between elements in $A$ and $B$.
- **Example:**

```
\begin{tikzpicture}
  \node (a) at (0,0) {a};
  \node (b) at (1,0) {b};
  \node (1) at (1,1) {1};
  \node (2) at (2,1) {2};
  \node (3) at (2,2) {3};
  \draw (a) -- (1);
  \draw (a) -- (2);
  \draw (b) -- (2);
  \draw (b) -- (3);
\end{tikzpicture}
```

- What is the difference between a **relation and a function from $A$ to $B$?**
Relations and functions

• Relations represent **one to many relationships** between elements in A and B.

• Example:

  ![Diagram of Relations](https://example.com/diagram.png)

• What is the difference between a relation and a function from A to B? A function defined on sets A, B assigns to each element in the domain set A exactly one element from B. So it is a special relation.

```plaintext
<table>
<thead>
<tr>
<th>A</th>
<th>1</th>
<th>2</th>
<th>3</th>
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</thead>
<tbody>
<tr>
<td>a</td>
<td></td>
<td></td>
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<tr>
<td>b</td>
<td></td>
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</tbody>
</table>
```

Relation on the set

**Definition:** A relation on the set A is a relation from A to itself.

**Example 1:**

• Let A = \{1,2,3,4\} and \(R_{div} = \{(a,b)| a \text{ divides } b\}\)

• What does \(R_{div}\) consist of?

• \(R_{div} = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}\)

```
<table>
<thead>
<tr>
<th>R</th>
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<th>2</th>
<th>3</th>
<th>4</th>
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<td>x</td>
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</tbody>
</table>
```
Relation on the set

Example:
- Let $A = \{1,2,3,4\}$.
- Define $a R b$ if and only if $a \neq b$.

$$R = \{(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,4),(4,1),(4,2),(4,3)\}$$

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<thead>
<tr>
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<td>x</td>
<td>x</td>
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</tbody>
</table>

Binary relations

- **Theorem:** The number of binary relations on a set $A$, where $|A| = n$ is: $2^{n^2}$

- **Proof:**
  - If $|A| = n$ then the cardinality of the Cartesian product $|A \times A| = n^2$.
  - $R$ is a binary relation on $A$ if $R \subseteq A \times A$ (that is, $R$ is a subset of $A \times A$).
  - The number of subsets of a set with $k$ elements: $2^k$
  - The number of subsets of $A \times A$ is: $2^{|A\times A|} = 2^{n^2}$
## Binary relations

- **Example:** Let $A = \{1, 2\}$
- What is $A \times A = \{(1,1), (1,2), (2,1), (2,2)\}$
- **List of possible relations (subsets of $A \times A$):**
  - $\emptyset$ .... 1
  - $\{(1,1)\}$, $\{(1,2)\}$, $\{(2,1)\}$, $\{(2,2)\}$ .... 4
  - $\{(1,1), (1,2)\}$, $\{(1,1), (2,1)\}$, $\{(1,1), (2,2)\}$ .... 6
  - $\{(1,2), (2,1)\}$, $\{(1,2), (2,2)\}$, $\{(2,1), (2,2)\}$ .... 6
  - $\{(1,1), (1,2), (2,1)\}$, $\{(1,1), (1,2), (2,2)\}$ .... 4
  - $\{(1,1), (2,1), (2,2)\}$, $\{(1,2), (2,1), (2,2)\}$ .... 4
  - $\{(1,1), (1,2), (2,1), (2,2)\}$ .... 1
- Use formula: $2^4 = 16$

## Properties of relations

**Definition (reflexive relation):** A relation $R$ on a set $A$ is called **reflexive** if $(a, a) \in R$ for every element $a \in A$.

**Example 1:**
- Assume relation $R_{\text{div}} = \{(a, b), \text{if } a \mid b\}$ on $A = \{1, 2, 3, 4\}$
- **Is $R_{\text{div}}$ reflexive?**
  - $R_{\text{div}} = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$
  - **Answer:** Yes. $(1,1)$, $(2,2)$, $(3,3)$, and $(4,4) \in A.$
Reflexive relation

- Reflexive relation
  - $R_{\text{div}} = \{(a, b), \text{ if } a | b\}$ on $A = \{1,2,3,4\}$
  - $R_{\text{div}} = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$

\[
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{array}
\]

- A relation $R$ is reflexive if and only if $MR$ has 1 in every position on its main diagonal.

---

Properties of relations

**Definition (reflexive relation)**: A relation $R$ on a set $A$ is called reflexive if $(a, a) \in R$ for every element $a \in A$.

**Example 2**:
- Relation $R_{\text{fun}}$ on $A = \{1,2,3,4\}$ defined as:
  - $R_{\text{fun}} = \{(1,2), (2,2), (3,3)\}$
- Is $R_{\text{fun}}$ reflexive?
  - No. It is not reflexive since $(1,1) \notin R_{\text{fun}}$. 

Properties of relations

**Definition (irreflexive relation):** A relation $R$ on a set $A$ is called **irreflexive** if $(a,a) \not\in R$ for every $a \in A$.

**Example 1:**
- Assume relation $R \neq$ on $A=\{1,2,3,4\}$, such that $a R \neq b$ if and only if $a \neq b$.
- Is $R \neq$ irreflexive?
- $R \neq = ...$
Irreflexive relation

Irreflexive relation
• \( R \neq \) on \( A = \{1,2,3,4\} \), such that \( a R \neq b \) if and only if \( a \neq b \).
• \( R_{\neq} = \{(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,4),(4,1),(4,2),(4,3)\} \)

<table>
<thead>
<tr>
<th></th>
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<tr>
<td>3</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

• A relation \( R \) is irreflexive if and only if \( MR \) has 0 in every position on its main diagonal.

Properties of relations

Definition (irreflexive relation): A relation \( R \) on a set \( A \) is called irreflexive if \( (a,a) \not\in R \) for every \( a \in A \).

Example 2:
• \( R_{\text{fun}} \) on \( A = \{1,2,3,4\} \) defined as:
  • \( R_{\text{fun}} = \{(1,2),(2,2),(3,3)\} \).
• Is \( R_{\text{fun}} \) irreflexive?
• Answer: No. Because \((2,2)\) and \((3,3)\) \( \not\in R_{\text{fun}} \)
Properties of relations

**Definition (symmetric relation):** A relation R on a set A is called **symmetric** if
\[ \forall a, b \in A \quad (a,b) \in R \rightarrow (b,a) \in R. \]

**Example 1:**
- \( R_{\text{div}} = \{(a,b), \text{ if } a \mid b\} \) on \( A = \{1,2,3,4\} \)
- **Is** \( R_{\text{div}} \) **symmetric?**
- \( R_{\text{div}} = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\} \)
- **Answer:** No. It is not symmetric since \((1,2) \in R \) but \((2,1) \notin R.\)

**Example 2:**
- \( R_{\neq} \) on \( A = \{1,2,3,4\} \), such that \( a R_{\neq} b \) if and only if \( a \neq b.\)
- **Is** \( R_{\neq} \) **symmetric?**
- \( R_{\neq} = \{(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,4),(4,1),(4,2),(4,3)\} \)
- **Answer:** Yes. If \((a,b) \in R_{\neq} \rightarrow (b,a) \in R_{\neq}.\)
Symmetric relation

**Symmetric relation:**
- $R \neq$ on $A = \{1,2,3,4\}$, such that $a \ R \neq b$ if and only if $a \neq b$.
- $R = \{(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,4),(4,1),(4,2),(4,3)\}$

\[
\begin{array}{cccc}
0 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 \\
MR & = & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 0 \\
\end{array}
\]

- **A relation $R$ is symmetric** if and only if $m_{ij} = m_{ji}$ for all $i,j$.

Properties of relations

**Definition (symmetric relation):** A relation $R$ on a set $A$ is called **symmetric** if
\[
\forall a, b \in A \quad (a,b) \in R \rightarrow (b,a) \in R.
\]

**Example 3:**
- Relation $R_{\text{fun}}$ on $A = \{1,2,3,4\}$ defined as:
  - $R_{\text{fun}} = \{(1,2),(2,2),(3,3)\}$.
- Is $R_{\text{fun}}$ symmetric?
- **Answer:** No. For $(1,2) \in R_{\text{fun}}$ there is no $(2,1) \in R_{\text{fun}}$