CS 441 Discrete Mathematics for CS
Lecture 9

Functions II

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Course administration

Midterm:
- **Wednesday, February 13, 2013**
- **Closed book, in-class**
- **Covers Chapters 1 and 2.1-2.3 of the textbook**

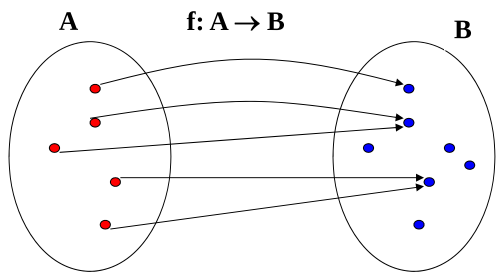
Homework 4 is out
- **Due on Monday, February 11, 2013**
- **Recitations today cover HW-4**

Course web page:

http://www.cs.pitt.edu/~milos/courses/cs441/

Functions

- **Definition**: Let A and B be two sets. A **function from A to B**, denoted **$f : A \to B$** , is an assignment of exactly one element of B to each element of A. We write f(a) = b to denote the assignment of b to an element a of A by the function f.



M. Hauskrecht

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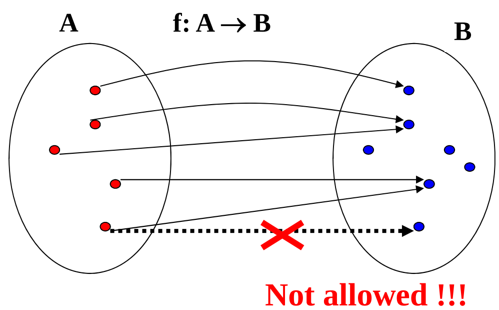

CS 441 Discrete mathematics for CS

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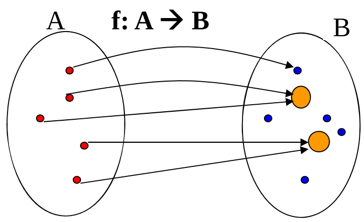
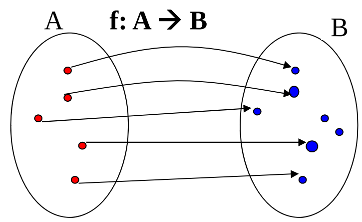
Injective function

Definition: A function f is said to be **one-to-one, or injective**, if and only if f(x) = f(y) implies x = y for all x, y in the domain of f. A function is said to be an **injection if it is one-to-one**.

Alternative: A function is one-to-one if and only if f(x) ≠ f(y), whenever x ≠ y. This is the contrapositive of the definition.



Not injective function

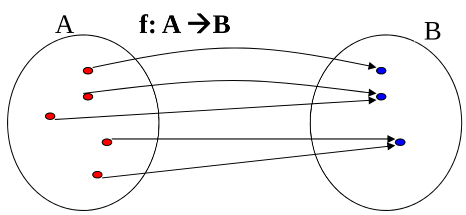


Injective function

Surjective function

Definition: A function f from A to B is called **onto**, or **surjective**, if and only if for every b ∈ B there is an element a ∈ A such that f(a) = b.

Alternative: all co-domain elements are covered



Surjective functions

Example 1: Let A = {1,2,3} and B = {a,b,c}

- Define f as
  - $1 \rightarrow c$
  - $2 \rightarrow a$
  - $3 \rightarrow c$

- Is f an onto?
- **No.** f is not onto, since $b \in B$ has no pre-image.

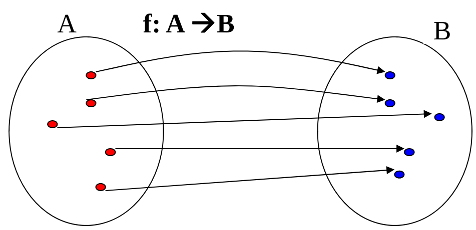
Example 2: A = {0,1,2,3,4,5,6,7,8,9}, B = {0,1,2}

- Define $h: A \rightarrow B$ as $h(x) = x \bmod 3$.

- Is h an onto function?
- **Yes.** h is onto since a pre-image of 0 is 6, a pre-image of 1 is 4, a pre-image of 2 is 8.

Bijective functions

Definition: A function f is called **a bijection** if it is **both one-to-one (injection) and onto (surjection)**.



Bijective functions

Example 1:

- Let A = {1,2,3} and B = {a,b,c}
  - Define f as
    - 1 → c
    - 2 → a
    - 3 → b
- Is f a bijection?
- **Yes.** It is both one-to-one and onto.

Bijective functions

Example 2:

- Define g : W → W (whole numbers), where
  g(n) = [n/2] (floor function).
    - 0 → [0/2] = [0] = 0
    - 1 → [1/2] = [1/2] = 0
    - 2 → [2/2] = [1] = 1
    - 3 → [3/2] = [3/2] = 1
- ...
- Is g a bijection?
  - **No.** g is onto but not 1-1 (g(0) = g(1) = 0 however 0 ≠ 1.

Bijective functions

- Let f be a function f: A → A from a set A to itself, where A is finite. Then f is one-to-one if and only if f is onto.

Assume

- **A is finite** and **f is one-to-one (injective)**
- Is f an **onto function (surjection)**?

Bijective functions

- Let f be a function f: A → A from a set A to itself, where A is finite. Then f is one-to-one if and only if f is onto.

Assume

- **A is finite** and **f is one-to-one (injective)**
- Is f an onto function (surjection)?
- **Yes.** Every element points to exactly one element. Injection assures they are different. So we have |A| different elements A points to. Since f: A → A the co-domain is covered thus the function is also a surjection (and a bijection)
- **A is finite and f is an onto function**
- Is the function one-to-one?

Bijective functions

- Let f be a function f: A → A from a set A to itself, where A is finite. Then f is one-to-one if and only if f is onto.

Assume

- **A is finite** and **f is one-to-one (injective)**
- Is it an onto function (surjection)?
- **Yes.** Every element points to exactly one element. Injection assures they are different. So we have |A| different elements A points to. Since f: A → A the co-domain is covered thus the function is also a surjection (and bijection)
- **A is finite and f is an onto function**
- Is the function one-to-one?
- **Yes.** Every element maps to exactly one element and all elements in A are covered. Thus the mapping must be one-to-one

Bijective functions

- Let f be a function from a set A to itself, where A is finite. Then f is one-to-one if and only if f is onto.

Please note the above is not true when A is an infinite set.

- **Example:**
  - f : Z → Z, where f(z) = 2 * z.
  - f is one-to-one but not onto.
    - 1 → **2**
    - 2 → **4**
    - 3 → **6**
  - 3 has no pre-image.

Functions on real numbers

Definition: Let f1 and f2 be functions from A to **R** (reals). Then f1 + f2 and f1 * f2 are also functions from A to **R** defined by

- (f1 + f2)(x) = f1(x) + f2(x)
- (f1 * f2)(x) = f1(x) * f2(x).

Examples:

- **Assume**
  - f1(x) = x - 1
  - f2(x) = x^3 + 1

  then
  - (f1 + f2)(x) = x^3 + x
  - (f1 * f2)(x) = x^4 - x^3 + x - 1.

Increasing and decreasing functions

Definition: A function f whose domain and codomain are subsets of real numbers is **strictly increasing** if f(x) > f(y) whenever x > y and x and y are in the domain of f. Similarly, f is called **strictly decreasing** if f(x) < f(y) whenever x > y and x and y are in the domain of f.

Note: Strictly increasing and strictly decreasing functions are one-to-one (injective).

Example:

- Let g : **R** $\rightarrow$ **R**, where g(x) = 2x - 1. Is it increasing ?
- **Proof .**

  For x>y holds 2x > 2y and subsequently 2x-1 > 2y-1

  Thus g is strictly increasing.

Identity function

Definition: Let A be a set. The **identity function** on A is the function i_A: A → A where $i_A(x) = x$.

Example:

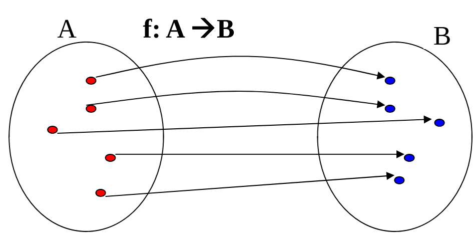
- Let A = {1,2,3}

Then:

- $i_A(1) = 1$
- $i_A(2) = 2$
- $i_A(3) = 3$.

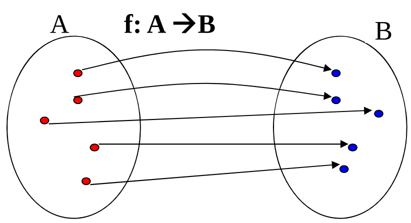
Bijective functions

Definition: A function f is called **a bijection** if it is **both one-to-one and onto**.

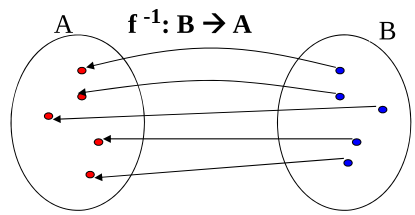


Inverse functions

Definition: Let f be a **bijection** from set A to set B. The **inverse function of f** is the function that assigns to an element b from B the unique element a in A such that f(a) = b. The inverse function of f is denoted by **f^{-1}**. Hence, f^{-1} (b) = a, when f(a) = b. If the inverse function of f exists, f is called **invertible.**



f is bijective



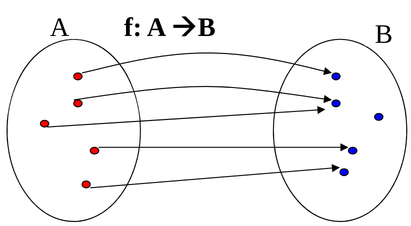
Inverse of f

Inverse functions

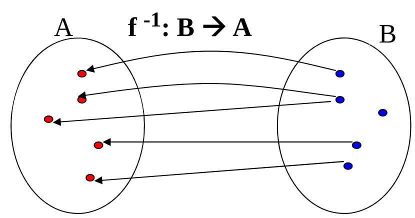
Note: if f is not a bijection then it is not possible to define the inverse function of f. **Why?**

Assume f is not one-to-one:

Inverse is not a function. One element of B is mapped to two different elements.



f



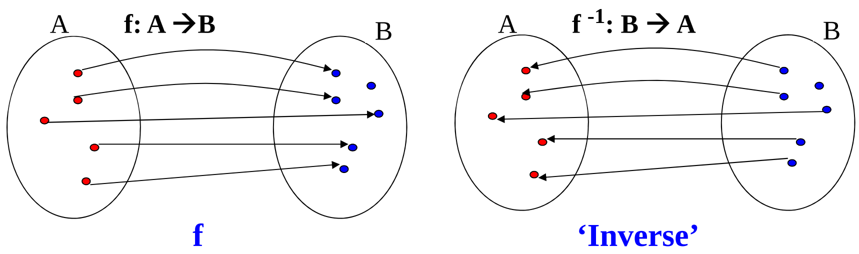
'Inverse'

Inverse functions

Note: if f is not a bijection then it is not possible to define the inverse function of f. Why?

Assume f is not onto:

Inverse is not a function. One element of B is not assigned any value in B.



Inverse functions

Example 1:

- Let A = {1,2,3} and i_A be the identity function

- $i_A(1) = 1$ $\quad$ $i_A^{-1}(1) = 1$
- $i_A(2) = 2$ $\quad$ $i_A^{-1}(2) = 2$
- $i_A(3) = 3$ $\quad$ $i_A^{-1}(3) = 3$

- Therefore, the inverse function of i_A is i_A.

Inverse functions

Example 2:

- Let g : **R** → **R**, where g(x) = 2x - 1.
- What is the inverse function g^{-1} ?

Approach to determine the inverse:

y = 2x - 1 => y + 1 = 2x

=> (y+1)/2 = x

- Define **g^{-1}(y) = x= (y+1)/2**

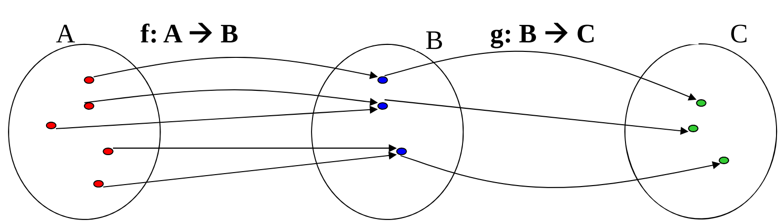
Test the correctness of inverse:

- **g(3)** = 2*3 - 1 = **5**
- **g^{-1} (5)** = (5+1)/2 = **3**
- g(**10**) = 2*10 - 1 = **19**
- g^{-1} (19) = (19+1)/2 = **10**.

Composition of functions

Definition: Let f be a function from set A to set B and let g be a function from set B to set C. The **composition of the functions g and f**, denoted by g ○ f is defined by

- **(g ○ f)(a) = g(f(a)).**



Composition of functions

Example 1:

- Let A = {1,2,3} and B = {a,b,c,d}

g : $A \to A$,	f: $A \to B$
$1 \to 3$	$1 \to b$
$2 \to 1$	$2 \to a$
$3 \to 2$	$3 \to d$

$f \circ g : A \to B$:

- $1 \to d$
- $2 \to b$
- $3 \to a$

Composition of functions

Example 2:

- Let f and g be two functions from Z to Z, where
- $f(x) = 2x$ and $g(x) = x^2$.
- $f \circ g : Z \to Z$
- $(f \circ g)(x) = f(g(x)) = f(x^2) = 2(x^2)$
- $g \circ f : Z \to Z$
- $(g \circ f)(x) = g(f(x)) = g(2x) = (2x)^2 = 4x^2$

Note that the order of the function composition matters

Composition of functions

Example 3:

- **$(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$, for all x.**
- Let $f : \mathbf{R} \rightarrow \mathbf{R}$, where $f(x) = 2x - 1$ and $f^{-1}(x) = (x+1)/2$.
- $(f \circ f^{-1})(x) = f(f^{-1}(x))$
  $= f((x+1)/2)$
  $= 2((x+1)/2) - 1$
  $= (x+1) - 1$
  $= x$
- $(f^{-1} \circ f)(x) = f^{-1}(f(x))$
  $= f^{-1}(2x - 1)$
  $= (2x)/2$
  $= x$

Some functions

Definitions:

- The **floor function** assigns a real number x the largest integer that is less than or equal to x. The floor function is denoted by $\lfloor x \rfloor$.
- The **ceiling function** assigns to the real number x the smallest integer that is greater than or equal to x. The ceiling function is denoted by $\lceil x \rceil$.

Other important functions:

- Factorials: $n! = n(n-1)!$ such that $1! = 1$