

CS 441 Discrete Mathematics for CS
Lecture 7

Sets and set operations

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Basic discrete structures

- **Discrete math** =
 - study of the discrete structures used to represent discrete objects
- Many discrete structures are built using [sets](#)
 - **Sets = collection of objects**

Examples of discrete structures built with the help of sets:

- **Combinations**
- **Relations**
- **Graphs**

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Set

- **Definition:** A **set** is a (unordered) collection of objects. These objects are sometimes called **elements** or **members** of the set. (Cantor's naive definition)
- **Examples:**
 - **Vowels in the English alphabet**
 $V = \{ a, e, i, o, u \}$
 - **First seven prime numbers.**
 $X = \{ 2, 3, 5, 7, 11, 13, 17 \}$

Representing sets

Representing a set by:

- 1) Listing (enumerating) the members of the set.
- 2) Definition by property, using the set builder notation
 $\{x \mid x \text{ has property } P\}$.

Example:

- Even integers between 50 and 63.
 - 1) $E = \{50, 52, 54, 56, 58, 60, 62\}$
 - 2) $E = \{x \mid 50 \leq x < 63, x \text{ is an even integer}\}$

If enumeration of the members is hard we often use ellipses.

Example: a set of integers between 1 and 100

- $A = \{1, 2, 3, \dots, 100\}$

Important sets in discrete math

- **Natural numbers:**
 - $\mathbf{N} = \{0, 1, 2, 3, \dots\}$
- **Integers**
 - $\mathbf{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$
- **Positive integers**
 - $\mathbf{Z}^+ = \{1, 2, 3, \dots\}$
- **Rational numbers**
 - $\mathbf{Q} = \{p/q \mid p \in \mathbf{Z}, q \in \mathbf{Z}, q \neq 0\}$
- **Real numbers**
 - $\mathbf{R}$

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Russell's paradox

Cantor's naive definition of sets leads to Russell's paradox:

- **Let $S = \{x \mid x \notin x\}$,**
is a set of sets that are not members of themselves.
- **Question:** Where does the set S belong to?
 - Is $S \in S$ or $S \notin S$?
- **Cases**
 - **$S \in S$?:** S does not satisfy the condition so it must hold that $S \notin S$ (or $S \in S$ does not hold)
 - **$S \notin S$?:** S is included in the set S and hence $S \notin S$ does not hold
- **A paradox:** we cannot decide if S belongs to S or not
- **Russell's answer:** theory of types – used for sets of sets

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Equality

Definition: Two sets are equal if and only if they have the same elements.

Example:

- $\{1,2,3\} = \{3,1,2\} = \{1,2,1,3,2\}$

Note: Duplicates don't contribute anything new to a set, so remove them. The order of the elements in a set doesn't contribute anything new.

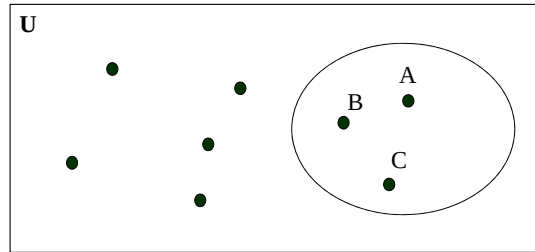
Example: Are $\{1,2,3,4\}$ and $\{1,2,2,4\}$ equal?
No!

Special sets

- **Special sets:**
 - The universal set is denoted by U : the set of all objects under the consideration.
 - The empty set is denoted as $\emptyset$ or $\{ \}$.

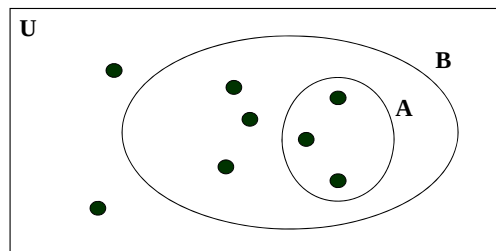
Venn diagrams

- A set can be visualized using **Venn Diagrams**:
 - $V = \{ A, B, C \}$



A Subset

- **Definition:** A set A is said to be a **subset** of B **if and only if** every element of A is also an element of B . We use $A \subseteq B$ to indicate **A is a subset of B** .



- Alternate way to define A is a subset of B :
$$\forall x (x \in A) \rightarrow (x \in B)$$

Empty set/Subset properties

Theorem $\emptyset \subseteq S$

- Empty set is a subset of any set.

Proof:

- Recall the definition of a subset: all elements of a set A must be also elements of B: $\forall x (x \in A \rightarrow x \in B)$.
- We must show the following implication holds for any S
 $\forall x (x \in \emptyset \rightarrow x \in S)$
- Since the empty set does not contain any element, $x \in \emptyset$ is **always False**
- Then the implication is **always True**.

End of proof

Subset properties

Theorem: $S \subseteq S$

- Any set S is a subset of itself

Proof:

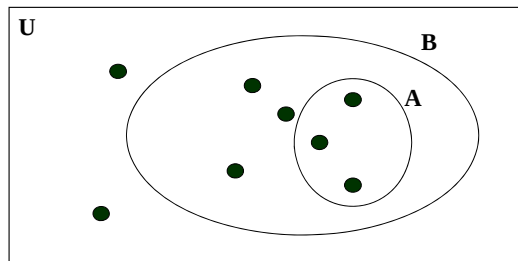
- the definition of a subset says: all elements of a set A must be also elements of B: $\forall x (x \in A \rightarrow x \in B)$.
- Applying this to S we get:
- $\forall x (x \in S \rightarrow x \in S)$ which is trivially **True**
- End of proof

Note on equivalence:

- Two sets are equal if each is a subset of the other set.

A proper subset

Definition: A set A is said to be a **proper subset** of B if and only if $A \subseteq B$ and $A \neq B$. We denote that A is a proper subset of B with the notation $A \subset B$.



Example: $A = \{1, 2, 3\}$ $B = \{1, 2, 3, 4, 5\}$

Is: $A \subset B$? Yes.

Cardinality

Definition: Let S be a set. If there are exactly n distinct elements in S , where n is a nonnegative integer, we say S is a finite set and that n is the **cardinality of S** . The cardinality of S is denoted by $|S|$.

Examples:

- $V = \{1, 2, 3, 4, 5\}$
 $|V| = 5$
- $A = \{1, 2, 3, 4, \dots, 20\}$
 $|A| = 20$
- $|\emptyset| = ?$

Cardinality

Definition: Let S be a set. If there are exactly n distinct elements in S , where n is a nonnegative integer, we say S is a finite set and that n is the **cardinality of S** . The cardinality of S is denoted by $|S|$.

Examples:

- $V = \{1, 2, 3, 4, 5\}$
 $|V| = 5$
- $A = \{1, 2, 3, 4, \dots, 20\}$
 $|A| = 20$
- $|\emptyset| = 0$

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Infinite set

Definition: A set is **infinite** if it is not finite.

Examples:

- The set of natural numbers is an infinite set.
- $\mathbb{N} = \{1, 2, 3, \dots\}$
- The set of reals is an infinite set.

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Power set

Definition: Given a set S , the **power set** of S is the set of all subsets of S . The power set is denoted by $P(S)$.

Examples:

- Assume an empty set $\emptyset$
- What is the power set of $\emptyset$? $P(\emptyset) = \{ \emptyset \}$
- What is the cardinality of $P(\emptyset)$? $|P(\emptyset)| = 1$.

- Assume set $\{1\}$
- $P(\{1\}) = \{ \emptyset, \{1\} \}$
- $|P(\{1\})| = 2$

Power set

- $P(\{1\}) = \{ \emptyset, \{1\} \}$
- $|P(\{1\})| = 2$

- Assume $\{1,2\}$
- $P(\{1,2\}) = \{ \emptyset, \{1\}, \{2\}, \{1,2\} \}$
- $|P(\{1,2\})| = 4$

- Assume $\{1,2,3\}$
- $P(\{1,2,3\}) = \{ \emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\} \}$
- $|P(\{1,2,3\})| = 8$

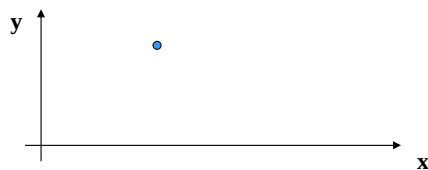
- **If S is a set with $|S| = n$ then $|P(S)| = 2^n$**

N-tuple

- Sets are used to represent unordered collections.
- **Ordered-n tuples** are used to represent an ordered collection.

Definition: An **ordered n-tuple** $(x_1, x_2, \dots, x_N)$ is the ordered collection that has x_1 as its first element, x_2 as its second element, ..., and x_N as its N -th element, $N \geq 2$.

Example:



- Coordinates of a point in the 2-D plane $(12, 16)$

Cartesian product

Definition: Let S and T be sets. The **Cartesian product of S and T** , denoted by **$S \times T$** , is the set of all ordered pairs (s,t) , where $s \in S$ and $t \in T$. Hence,

$$S \times T = \{ (s,t) \mid s \in S \wedge t \in T \}.$$

Examples:

- $S = \{1,2\}$ and $T = \{a,b,c\}$
- $S \times T = \{ (1,a), (1,b), (1,c), (2,a), (2,b), (2,c) \}$
- $T \times S = \{ (a,1), (a,2), (b,1), (b,2), (c,1), (c,2) \}$
- Note: $S \times T \neq T \times S$!!!!

Cardinality of the Cartesian product

- $|S \times T| = |S| * |T|$.

Example:

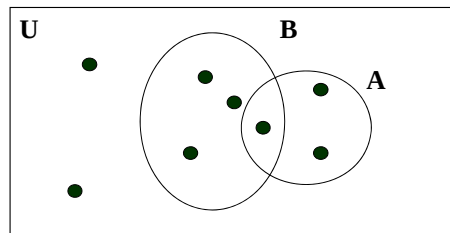
- $A = \{\text{John, Peter, Mike}\}$
- $B = \{\text{Jane, Ann, Laura}\}$
- $A \times B = \{(\text{John, Jane}), (\text{John, Ann}), (\text{John, Laura}), (\text{Peter, Jane}), (\text{Peter, Ann}), (\text{Peter, Laura}), (\text{Mike, Jane}), (\text{Mike, Ann}), (\text{Mike, Laura})\}$
- $|A \times B| = 9$
- $|A|=3, |B|=3 \rightarrow |A| |B|= 9$

Definition: A subset of the Cartesian product $A \times B$ is called a relation from the set A to the set B .

Set operations

Definition: Let A and B be sets. The **union of A and B** , denoted by $A \cup B$, is the set that contains those elements that are either in A or in B , or in both.

- Alternate: $A \cup B = \{ x \mid x \in A \vee x \in B \}$.



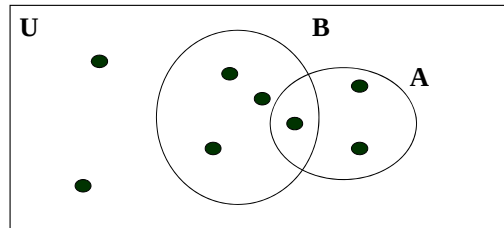
Example:

- $A = \{1,2,3,6\}$ $B = \{2,4,6,9\}$
- $A \cup B = \{1,2,3,4,6,9\}$

Set operations

Definition: Let A and B be sets. The **intersection of A and B**, denoted by $A \cap B$, is the set that contains those elements that are in both A and B.

- Alternate: $A \cap B = \{ x \mid x \in A \wedge x \in B \}$.



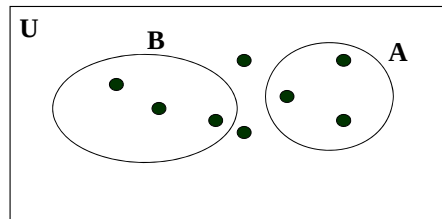
Example:

- $A = \{1, 2, 3, 6\}$ $B = \{2, 4, 6, 9\}$
- $A \cap B = \{2, 6\}$

Disjoint sets

Definition: Two sets are called **disjoint** if their intersection is empty.

- Alternate: A and B are disjoint **if and only if** $A \cap B = \emptyset$.



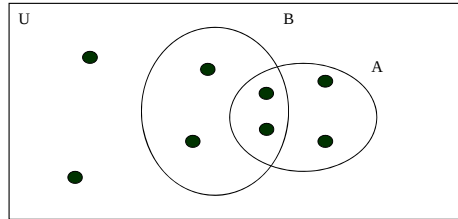
Example:

- $A = \{1, 2, 3, 6\}$ $B = \{4, 7, 8\}$ Are these disjoint?
- Yes.
- $A \cap B = \emptyset$

Cardinality of the set union

Cardinality of the set union.

- $|A \cup B| = |A| + |B| - |A \cap B|$



- Why this formula? Correct for an over-count.
- More general rule:
 - **The principle of inclusion and exclusion.**