## CS 441 Discrete Mathematics for CS
## Lecture 6

# Formal and informal proofs

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## Announcements

- **Homework assignment 2 due today**
- **Homework assignment 3:**
  - **posted on the course web page**
  - **Due on Monday February 4, 2013**
- **Recitations on Wednesday:**
  - **Practice problems related to assignment 3**

## Theorems and proofs

- The truth value of some statement about the world is obvious and easy to assign
- The truth of other statements may not be obvious, …

  …. But it may still follow (be derived) from known facts about the world

To show the truth value of such a statement following from other statements we need to provide **a correct supporting argument**

- **a proof**

**Important questions:**

- When is the argument correct?
- How to construct a correct argument, what method to use?



## Theorems and proofs

- **Theorem:** a statement that can be shown to be true.
  - **Typically the theorem looks like this:**

    $(p1 \wedge p2 \wedge p3 \wedge \ldots \wedge pn) \rightarrow q$

    **Premises** (p1 ∧ … ∧ pn), **conclusion** (q)

- **Example:**

  Fermat's Little theorem:
  - If p is a prime and a is an integer not divisible by p, then: $a^{p-1} \equiv 1 \mod p$

## Theorems and proofs

- **Theorem:** a statement that can be shown to be true.
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    $(p1 \wedge p2 \wedge p3 \wedge \ldots \wedge pn) \rightarrow q$

    **Premises** **conclusion**

- **Example:**

  Fermat's Little theorem:

  **Premises (hypotheses)**

  - If p is a prime and a is an integer not divisible by p, then: $a^{p-1} \equiv 1 \bmod p$

  **conclusion**



## Formal proofs

**Proof:**

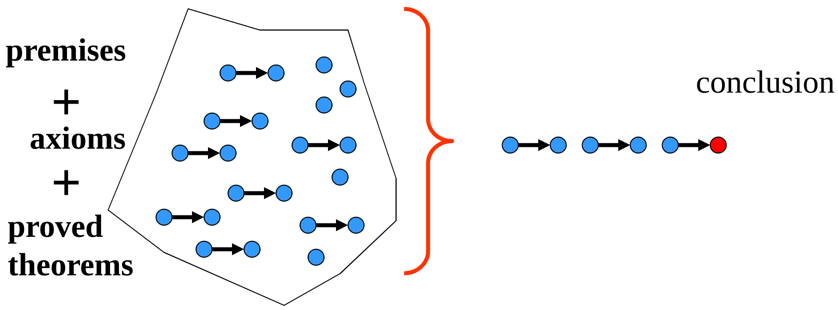
- Provides an argument supporting the validity of the statement
- Proof of the theorem:
  - shows that the conclusion follows from premises
  - may use:
    - Premises
    - Axioms
    - Results of other theorems

**Formal proofs:**

- steps of the proofs **follow logically** from the set of premises and axioms

# Formal proofs

- **Formal proofs:**
  - show that **steps of the proofs follow logically** from the set of hypotheses and axioms




**In this class we assume formal proofs in the propositional logic**



# Special case: equivalences

**Proofs based on logical equivalences.** A proposition or its part can be transformed using a sequence of equivalence rewrites till some conclusion can be reached. **Important:** Equivalences rewrite propositions whether they are True of False.

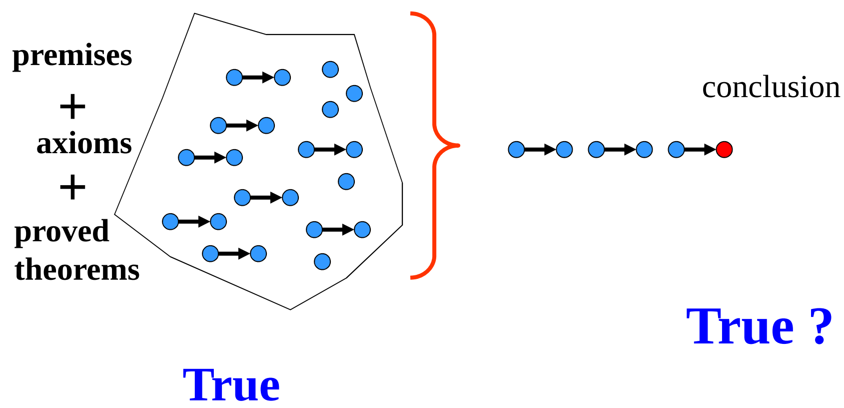
**Example:** Show that $(p \wedge q) \rightarrow p$ is a tautology.

- Proof: (we must show $(p \wedge q) \rightarrow p \iff T$)

| | | | |
|---|---|---|---|
| $(p \wedge q) \rightarrow p$ | <=> | $\neg(p \wedge q) \vee p$ | Useful |
| | <=> | $[\neg p \vee \neg q] \vee p$ | DeMorgan |
| | <=> | $[\neg q \vee \neg p] \vee p$ | Commutative |
| | <=> | $\neg q \vee [\neg p \vee p]$ | Associative |
| | <=> | $\neg q \vee [T]$ | Useful |

# General proofs

- Infer new **True statements from known True statements**






# Rules of inference

**Rules of inference:**

- **Allow us to infer from True statements new True statements**
- **Represent logically valid inference patterns**

**Example:**

- **Modus Ponens**, or the Law of Detachment
- Rule of inference

$$\begin{array}{l} p \\ \underline{p \rightarrow q} \\ \therefore\ q \end{array}$$

- Given p is true and the implication $p \rightarrow q$ is true then q is true.

## Rules of inference

**Rules of inference: logically valid inference patterns**

**Example;**

- **Modus Ponens**, or the Law of Detachment
- Rule of inference

$$\begin{array}{l} p \\ \underline{p \rightarrow q} \\ \therefore q \end{array}$$

- Given p is true and the implication $p \rightarrow q$ is true then q is true.

| p | q | $p \rightarrow q$ |
|---|---|---|
| *False* | *False* | *True* |
| *False* | *True* | *True* |
| *True* | *False* | *False* |
| *True* | *True* | *True* |

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## Rules of inference

**Rules of inference: logically valid inference patterns**

**Example:**

- **Modus Ponens**, or the Law of Detachment
- Rules of inference

$$\begin{array}{l} p \\ \underline{p \rightarrow q} \\ \therefore q \end{array}$$

- Given p is true and the implication $p \rightarrow q$ is true then q is true.
- **Tautology Form:** $(p \wedge (p \rightarrow q)) \rightarrow q$

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## Rules of inference

- **Addition**

  $p \rightarrow (p \lor q)$

  $$\frac{p}{\therefore\ p \lor q}$$

- **Example:** It is below freezing now. Therefore, it is below freezing or raining snow.

- **Simplification**

  $(p \land q) \rightarrow p$

  $$\frac{p \land q}{\therefore\ p}$$

- **Example:** It is below freezing and snowing. Therefore it is below freezing.



## Rules of inference

- **Modus Tollens (modus ponens for the contrapositive)**

  $[\neg q \land (p \rightarrow q)] \rightarrow \neg p$

  $$\begin{array}{l} \neg q \\ \underline{p \rightarrow q} \\ \therefore\ \neg p \end{array}$$

- **Hypothetical Syllogism**

  $[(p \rightarrow q) \land (q \rightarrow r)] \rightarrow (p \rightarrow r)$

  $$\begin{array}{l} p \rightarrow q \\ \underline{q \rightarrow r} \\ \therefore\ p \rightarrow r \end{array}$$

- **Disjunctive Syllogism**

  $[(p \lor q) \land \neg p] \rightarrow q$

  $$\begin{array}{l} p \lor q \\ \underline{\neg p} \\ \therefore\ q \end{array}$$

## Rules of inference

- **Logical equivalences (discussed earlier)**

  **A <=> B**

  **$A \rightarrow B$ is a tautology**

**Example: De Morgan Law**

$\neg( p \vee q ) \iff \neg p \wedge \neg q$

$\neg( p \vee q ) \rightarrow \neg p \wedge \neg q$ **is a tautology**



## Rules of inference

- A **valid argument** is one built using the rules of inference from premises (hypotheses). When all premises are true the argument should lead us to the correct conclusion.
- $(p1 \wedge p2 \wedge p3 \wedge \ldots \wedge pn ) \rightarrow q$
- **How to use the rules of inference?**

## Applying rules of inference

**Assume** the following statements (hypotheses):

- It is not sunny this afternoon and it is colder than yesterday.
- We will go swimming only if it is sunny.
- If we do not go swimming then we will take a canoe trip.
- If we take a canoe trip, then we will be home by sunset.

**Show** that all these lead to a conclusion:

- We will be home by sunset.



## Applying rules of inference

**Text:**

(1) It is not sunny this afternoon and it is colder than yesterday.

(2) We will go swimming only if it is sunny.

(3) If we do not go swimming then we will take a canoe trip.

(4) If we take a canoe trip, then we will be home by sunset.

**Propositions:**

- p = It is sunny this afternoon, q = it is colder than yesterday, r = We will go swimming , s= we will take a canoe trip
- t= We will be home by sunset

**Translation:**

- **Assumptions:** (1) $\neg p \wedge q$, (2) $r \rightarrow p$, (3) $\neg r \rightarrow s$, (4) $s \rightarrow t$
- **We want to show:** t

## Proofs using rules of inference

**Translations:**

- **Assumptions:** $\neg p \wedge q$, $r \rightarrow p$, $\neg r \rightarrow s$, $s \rightarrow t$
- **We want to show:** $t$

**Proof:**

- 1. $\neg p \wedge q$ Hypothesis
- 2. $\neg p$ Simplification
- 3. $r \rightarrow p$ Hypothesis
- 4. $\neg r$ Modus tollens (step 2 and 3)
- 5. $\neg r \rightarrow s$ Hypothesis
- 6. $s$ Modus ponens (steps 4 and 5)
- 7. $s \rightarrow t$ Hypothesis
- 8. $t$ Modus ponens (steps 6 and 7)
- **end of proof**



## Informal proofs

**Proving theorems in practice:**

- The steps of the proofs are not expressed in any formal language as e.g. propositional logic
- **Steps are argued less formally** using English, mathematical formulas and so on
- One must always watch the consistency of the argument made, logic and its rules can often help us to decide the soundness of the argument if it is in question

- **We use (informal) proofs to illustrate different methods of proving theorems**

# Methods of proving theorems

**Basic methods to prove the theorems:**

- **Direct proof**
  - $p \rightarrow q$ is proved by showing that if p is true then q follows
- **Indirect proof**
  - Show the contrapositive $\neg q \rightarrow \neg p$. If $\neg q$ holds then $\neg p$ follows
- **Proof by contradiction**
  - Show that $(p \wedge \neg q)$ contradicts the assumptions
- **Proof by cases**
- **Proofs of equivalence**
  - $p \leftrightarrow q$ is replaced with $(p \rightarrow q) \wedge (q \rightarrow p)$

Sometimes one method of proof does not go through as nicely as the other method. You may need to try more than one approach.



# Direct proof

- $p \rightarrow q$ is proved by showing that if p is true then q follows
- **Example:** Prove that "If n is odd, then $n^2$ is odd."

**Proof:**

- Assume the premise (hypothesis) is true, i.e. suppose n is odd.
- Then $n = 2k + 1$, where k is an integer.

## Direct proof

- $p \rightarrow q$ is proved by showing that if p is true then q follows
- **Example:** Prove that "If n is odd, then $n^2$ is odd."

**Proof:**

- Assume the hypothesis is true, i.e. suppose n is odd.
- Then $n = 2k + 1$, where k is an integer.

$$n^2 = (2k + 1)^2$$
$$= 4k^2 + 4k + 1$$
$$= 2(2k^2 + 2k) + 1$$



## Direct proof

- $p \rightarrow q$ is proved by showing that if p is true then q follows
- **Example:** Prove that "If n is odd, then $n^2$ is odd."

**Proof:**

- Assume the hypothesis is true, i.e. suppose n is odd.
- Then $n = 2k + 1$, where k is an integer.

$$n^2 = (2k + 1)^2$$
$$= 4k^2 + 4k + 1$$
$$= 2(2k^2 + 2k) + 1$$

- Therefore, $n^2$ is odd. □

## Direct proof

- Direct proof may not be the best option. It may become hard to prove the conclusion follows from the premises.

**Example:** Prove If 3n + 2 is odd then **n is odd**.

**Proof:**

- Assume that 3n + 2 is odd,
  - thus 3n + 2 = 2k + 1 for some k.
- Then n = (2k – 1)/3
- ?



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**Example:** Prove If 3n + 2 is odd then **n is odd**.

**Proof:**

- Assume that 3n + 2 is odd,
  - thus 3n + 2 = 2k + 1 for some k.
- Then n = (2k – 1)/3
- **Not clear how to continue**

## Indirect proof

- To show $p \rightarrow q$ prove its contrapositive $\neg q \rightarrow \neg p$
- Why? **$p \rightarrow q$ and $\neg q \rightarrow \neg p$ are equivalent !!!**
- Assume $\neg q$ is true, show that $\neg p$ is true.

**Example:** Prove If $3n + 2$ is odd then n is odd.

**Proof:**



## Indirect proof

- To show $p \rightarrow q$ prove its contrapositive $\neg q \rightarrow \neg p$
- Why? **$p \rightarrow q$ and $\neg q \rightarrow \neg p$ are equivalent !!!**
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**Example:** Prove If $3n + 2$ is odd then **n is odd**.

**Proof:**

- Assume **n is even**, that is $n = 2k$, where k is an integer.

## Indirect proof

- To show $p \rightarrow q$ prove its contrapositive $\neg q \rightarrow \neg p$
- Why? **$p \rightarrow q$ and $\neg q \rightarrow \neg p$ are equivalent !!!**
- Assume $\neg q$ is true, show that $\neg p$ is true.

**Example:** Prove If 3n + 2 is odd then n is odd.

**Proof:**

- Assume n is even, that is n = 2k, where k is an integer.
- Then: $3n + 2 = 3(2k) + 2$
  $= 6k + 2$
  $= 2(3k+1)$



## Indirect proof

- To show $p \rightarrow q$ prove its contrapositive $\neg q \rightarrow \neg p$
- Why? **$p \rightarrow q$ and $\neg q \rightarrow \neg p$ are equivalent !!!**
- Assume $\neg q$ is true, show that $\neg p$ is true.

**Example:** Prove If **3n + 2 is odd** then n is odd.

**Proof:**

- Assume n is even, that is n = 2k, where k is an integer.
- Then: $3n + 2 = 3(2k) + 2$
  $= 6k + 2$
  $= 2(3k+1)$
- Therefore **3n + 2 is even.**

## Indirect proof

- To show $p \rightarrow q$ prove its contrapositive $\neg q \rightarrow \neg p$
- Why? **$p \rightarrow q$ and $\neg q \rightarrow \neg p$ are equivalent !!!**
- Assume $\neg q$ is true, show that $\neg p$ is true.

**Example:** Prove If $3n + 2$ is odd then n is odd.

**Proof:**

- Assume n is even, that is $n = 2k$, where k is an integer.
- Then: $3n + 2 = 3(2k) + 2$
  $= 6k + 2$
  $= 2(3k+1)$
- Therefore $3n + 2$ is even.
- We proved **¬ "n is odd" → ¬ "3n + 2 is odd".** This is equivalent to **"3n + 2 is odd" → "n is odd".** □



## Proof by contradiction

- We want to prove $p \rightarrow q$
- The only way to reject (or disprove) $p \rightarrow q$ is to show that $(p \wedge \neg q)$ can be true
- However, if we manage to prove that either q or ¬ p is True then we contradict **$(p \wedge \neg q)$**
  - **and subsequently $p \rightarrow q$** must be true
- Proof by contradiction. Show that the assumption **$(p \wedge \neg q)$ leads either to** q or ¬ p which generates a contradiction.

## Proof by contradiction

- We want to prove $p \rightarrow q$
- To reject **$p \rightarrow q$** show that **$(p \wedge \neg q)$** can be true
- To reject **$(p \wedge \neg q)$** show that either **$q$ or $\neg p$** is True

**Example:** Prove If **3n + 2 is odd** then **n is odd**.

**Proof:**

- Assume **3n + 2 is odd** and **n is even**, that is n = 2k, where k an integer.



## Proof by contradiction

- We want to prove $p \rightarrow q$
- To reject **$p \rightarrow q$** show that **$(p \wedge \neg q)$** can be true
- To reject **$(p \wedge \neg q)$** show that either **$q$ or $\neg p$** is True

**Example:** Prove If **3n + 2 is odd** then **n is odd.**

**Proof:**

- Assume **3n + 2 is odd** and **n is even**, that is n = 2k, where k an integer.
- Then: $3n + 2 = 3(2k) + 2$
  $= 6k + 2$
  $= 2(3k + 1)$
- Thus **3n + 2 is…**

## Proof by contradiction

- We want to prove $p \rightarrow q$
- To reject **$p \rightarrow q$** show that **$(p \wedge \neg q)$** can be true
- To reject **$(p \wedge \neg q)$** show that either **$q$ or $\neg p$** is True

**Example:** Prove If **3n + 2 is odd** then n is odd.

**Proof:**

- Assume **3n + 2 is odd** and n is even, that is n = 2k, where k an integer.
- Then: $3n + 2 = 3(2k) + 2$
  $= 6k + 2$
  $= 2(3k + 1)$
- Thus **3n + 2 is even.** This is a contradiction with the assumption that **3n + 2 is odd.** Therefore **n is odd**. □



## Vacuous proof

**We want to show** $p \rightarrow q$

- Suppose p (the hypothesis) is always false
- Then $p \rightarrow q$ is always true.

**Reason:**

- $F \rightarrow q$ is always T, whether q is True or False

**Example:**

- Let P(n) denotes "if $n > 1$ then $n^2 > n$" is TRUE.
- Show that P(0).

**Proof:**

- For n=0 the premise is False. Thus P(0) is always true.

## Trivial proofs

**We want to show** $p \rightarrow q$

- Suppose the conclusion q is always true
- Then the implication $p \rightarrow q$ is trivially true.
- **Reason:**
- $p \rightarrow T$ is always T, whether p is True or False

**Example:**

- Let P(n) is "if a >=b then $a^n >= b^n$"
- Show that P(0)

**Proof:**

$a^0 >= b^0$ is 1=1 trivially true.



## Proof by cases

- We want to show $p1 \vee p2 \vee \ldots \vee pn \rightarrow q$
- Note that this is equivalent to

  $(p1 \rightarrow q) \wedge (p2 \rightarrow q) \wedge \ldots \wedge (pn \rightarrow q)$
- **Why?**

## Proof by cases

- We want to show $p1 \vee p2 \vee \ldots \vee pn \rightarrow q$
- Note that this is equivalent to
  - $(p1 \rightarrow q) \wedge (p2 \rightarrow q) \wedge \ldots \wedge (pn \rightarrow q)$
- **Why?**
- $p1 \vee p2 \vee \ldots \vee pn \rightarrow q$ <=> (useful)
- $\neg (p1 \vee p2 \vee \ldots \vee pn) \vee q$ <=> (De Morgan)
- $(\neg p1 \wedge \neg p2 \wedge \ldots \wedge \neg pn) \vee q$ <=> (distributive)
- $(\neg p1 \vee q) \wedge (\neg p2 \vee q) \wedge \ldots \wedge (\neg pn \vee q)$ <=> (useful)
- $(p1 \rightarrow q) \wedge (p2 \rightarrow q) \wedge \ldots \wedge (pn \rightarrow q)$



## Proof by cases

We want to show $p1 \vee p2 \vee \ldots \vee pn \rightarrow q$

- Equivalent to $(p1 \rightarrow q) \wedge (p2 \rightarrow q) \wedge \ldots \wedge (pn \rightarrow q)$

**Prove individual cases as before. All of them must be true.**

**Example:** Show that $|x||y|=|xy|$.

**Proof:**

- 4 cases:
- $x >= 0, y >= 0$ $xy > 0$ and $|xy|=xy=|x||y|$
- $x >= 0, y < 0$ $xy < 0$ and $|xy|=-xy=x(-y)=|x||y|$
- $x < 0, y >= 0$ $xy < 0$ and $|xy|=-xy=(-x)y=|x||y|$
- $x < 0, , y < 0$ $xy > 0$ and $|xy|=(-x)(-y)=|x||y|$
- All cases proved.

## Proof of equivalences

**We want to prove $p \leftrightarrow q$**

- Statements: p if and only if q.
- Note that $p \leftrightarrow q$ is equivalent to $[ (p \rightarrow q) \wedge (q \rightarrow p) ]$
- Both implications must hold.

**Example:**

- Integer is odd if and only if n^2 is odd.

**Proof of $(p \rightarrow q)$ :**

- **$(p \rightarrow q)$** If n is odd then n^2 is odd
- we use a direct proof
- Suppose n is odd. Then n = 2k + 1,where k is an integer.
- n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1
- Therefore, n^2 is odd.

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## Proof of equivalences

**We want to prove $p \leftrightarrow q$**

- Note that $p \leftrightarrow q$ is equivalent to $[ (p \rightarrow q) \wedge (q \rightarrow p) ]$
- Both implications must hold.

- Integer is odd if and only if n^2 is odd.

**Proof of** $(q \rightarrow p)$**:**

- $(q \rightarrow p)$: if n^2 is odd then n is odd
- we use an indirect proof $(\neg p \rightarrow \neg q)$ is a contrapositive
- n is even that is n = 2k,
- then n^2 = 4k^2= 2(2k^2)
- Therefore n^2 is even. Done proving the contrapositive.

**Since both $(p \rightarrow q)$ and $(q \rightarrow p)$ are true the equivalence is true**

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## Proofs with quantifiers

- **Existence proof**
  - **Constructive**
    - Find the example that shows the statement holds.
  - **Nonconstructive**
    - Show it holds for one example but we do not have the witness example (typically ends with one example or other example)

- **Counterexamples:**
  - use to disprove a universal statements