CS 441 Discrete Mathematics for CS
Lecture 5

Predicate logic

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Announcements

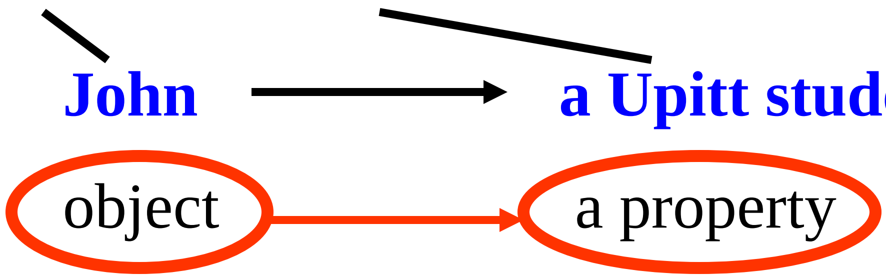
- Homework assignment 1 due today
- Homework assignment 2:
  - posted on the course web page
  - Due on Monday January 28, 2013
- Recitations today:
  - Practice problems related to assignment 2

Propositional logic: limitations

Propositional logic: the world is described in terms of elementary propositions and their logical combinations

Elementary statements:

- **Typically refer to objects, their properties and relations. But these are not explicitly represented** in the propositional logic
  - **Example:**
    - "John is a UPitt student."



    - Objects and properties are hidden in the statement, it is not possible to reason about them

Propositional logic: limitations

(1) Statements that hold for many objects must be enumerated

- **Example:**
  - John is a CS UPitt graduate → John has passed cs441
  - Ann is a CS Upitt graduate → Ann has passed cs441
  - Ken is a CS Upitt graduate → Ken has passed cs441
  - ...
- **Solution:** make statements with **variables**
  - x is a CS UPitt graduate → x has passed cs441

Propositional logic: limitations

(2) Statements that define the property of the group of objects

- **Example:**
  - All new cars must be registered.
  - Some of the CS graduates graduate with honor.

- **Solution:** make statements with **quantifiers**
  - **Universal quantifier** –the property is satisfied by all members of the group
  - **Existential quantifier** – at least one member of the group satisfy the property

Predicate logic

Remedies the limitations of the propositional logic

- Explicitly models objects and their properties
- Allows to make statements with variables and quantify them

Predicate logic:

- **Constant** –models a specific object

  Examples: "John", "France", "7"

- **Variable** – represents object of specific type **(defined by the *universe of discourse*)**

  Examples: x, y

  (universe of discourse can be people, students, numbers)

- **Predicate** **-** over one, two or many variables or constants.
  - Represents properties or relations among objects

  Examples: Red(car23), student(x), married(John,Ann)

Predicates

Predicates represent properties or relations among objects

- A predicate P(x) assigns a value **true or false** to each x depending on whether the property holds or not for x.
- The assignment is best viewed as a big table with the variable x substituted for objects from ***the universe of discourse***

Example:

- Assume **Student(x)** where the universe of discourse are people
- Student(John) …. T (if John is a student)
- Student(Ann) …. T (if Ann is a student)
- Student(Jane) ….. F (if Jane is not a student)
- …

Predicates

Assume a predicate P(x) that represents the statement:

- **x is a prime number**

Truth values for different x:

- P(2) T
- P(3) T
- P(4) F
- P(5) T
- P(6) F

All statements P(2), P(3), P(4), P(5), P(6) are propositions

…

But P(x) with variable x is not a proposition

Compound statements in predicate logic

Compound statements are obtained via logical connectives

Examples:

Student(Ann) $\wedge$ Student(Jane)

- **Translation:** “Both Ann and Jane are students”
- **Proposition:** yes.

Country(Sienna) $\vee$ River(Sienna)

- **Translation:** “Sienna is a country or a river”
- **Proposition:** yes.

CS-major(x) $\rightarrow$ Student(x)

- **Translation:** “if x is a CS-major then x is a student”
- **Proposition:** no.

Quantified statements

Predicate logic lets us to make statements about groups of objects

- To do this we use **special quantified expressions**

Two types of quantified statements:

- **universal**

  Example: ‘ all CS Upitt graduates have to pass cs441”

  – the statement is true for all graduates

- **existential**

  Example: ‘Some CS Upitt students graduate with honor.’

  – the statement is true for some people

Universal quantifier

Quantification converts a propositional function into **a proposition** by binding a variable to a set of values from the universe of discourse.

Example:

- Let P(x) denote x > x - 1.
- Is P(x) a proposition? **No.** Many possible substitutions.
- Is $\forall x$ P(x) a proposition? **Yes.** True if for all x from the universe of discourse P(x) is true.

Universally quantified statements

Predicate logic lets us make statements about groups of objects

Universally quantified statement

- CS-major(x) → Student(x)
  - **Translation:** "if x is a CS-major then x is a student"
  - **Proposition:** **no.**
- $\forall$x CS-major(x) → Student(x)
  - **Translation:** "(For all people it holds that) if a person is a CS-major then she is a student."
  - **Proposition:** **yes.**

Existentially quantified statements

Statements about groups of objects

Example:

- CS-Upitt-graduate (x) $\wedge$ Honor-student(x)
  - **Translation:** "x is a CS-Upitt-graduate and x is an honor student"
  - **Proposition: no.**

- $\exists$ x CS-Upitt-graduate (x) $\wedge$ Honor-student(x)
  - **Translation:** "There is a person who is a CS-Upitt-graduate and who is also an honor student."
  - **Proposition: yes**.

Summary of quantified statements

- When **∀x P(x)** and **∃x P(x)** are true and false?

Statement	When true?	When false?
$\forall x\, P(x)$	P(x) true for all x	There is an x where P(x) is false.
$\exists x\, P(x)$	There is some x for which P(x) is true.	P(x) is false for all x.

Suppose the elements in the universe of discourse can be enumerated as x1, x2, ..., xN then:

- **$\forall x\, P(x)$ is true whenever $P(x1) \wedge P(x2) \wedge \ldots \wedge P(xN)$ is true**
- **$\exists x\, P(x)$ is true whenever $P(x1) \vee P(x2) \vee \ldots \vee P(xN)$ is true**.

Translation with quantifiers

Sentence:

- All Upitt students are smart.

- **Assume:** the domain of discourse of x are Upitt students
- **Translation:**
- $\forall x$ Smart(x)

- **Assume:** the universe of discourse are students (all students):
- $\forall x$ at(x,Upitt) $\rightarrow$ Smart(x)

- **Assume:** the universe of discourse are people:
- $\forall x$ student(x) $\wedge$ at(x,Upitt) $\rightarrow$ Smart(x)

Translation with quantifiers

Sentence:

- Someone at CMU is smart.

- **Assume:** the domain of discourse are all CMU affiliates
- **Translation:**
- $\exists$ x Smart(x)

- **Assume:** the universe of discourse are people:
- $\exists$ x at(x,CMU) $\wedge$ Smart(x)

Translation with quantifiers

- **Assume two predicates S(x) and P(x)**

Universal statements typically tie with implications

- **All S(x) is P(x)**
  - $\forall x\ (S(x) \rightarrow P(x))$
- **No S(x) is P(x)**
  - $\forall x(S(x) \rightarrow \neg P(x))$

Existential statements typically tie with conjunctions

- **Some S(x) is P(x)**
  - $\exists x\ (S(x) \wedge P(x))$
- **Some S(x) is not P(x)**
  - $\exists x\ (S(x) \wedge \neg P(x))$

Nested quantifiers

- More than one quantifier may be necessary to capture the meaning of a statement in the predicate logic.

Example:

- Every real number has its corresponding negative.
- **Translation:**
  - Assume:
    - a real number is denoted as x and its negative as y
    - A predicate P(x,y) denotes: "x + y =0"
- Then we can write:

$\forall x \, \exists y \, P(x,y)$

Nested quantifiers

- More than one quantifier may be necessary to capture the meaning of a statement in the predicate logic.

Example:

- There is a person who loves everybody.
- **Translation:**
  - Assume:
    - Variables x and y denote people
    - A predicate L(x,y) denotes: “x loves y”
- Then we can write in the predicate logic:

  $\exists x \, \forall y \;\; L(x,y)$

Order of quantifiers

The order of nested quantifiers matters if quantifiers are of different type

- $\forall x \exists y\, L(x,y)$ is not the same as $\exists y \forall x\, L(x,y)$

Example:

- Assume L(x,y) denotes "x loves y"

- Then: $\forall x \exists y\, L(x,y)$
- Translates to: Everybody loves somebody.
- And: $\exists y\, \forall x\, L(x,y)$
- Translates to: There is someone who is loved by everyone.

The meaning of the two is different.

Order of quantifiers

The order of nested quantifiers does not matter if quantifiers are of the same type

Example:

- For all x and y, if x is a parent of y then y is a child of x
- **Assume:**
  - Parent(x,y) denotes “x is a parent of y”
  - Child(x,y) denotes “x is a child of y”

- Two equivalent ways to represent the statement:
  - $\forall x \, \forall y \; Parent(x,y) \rightarrow Child(y,x)$
  - $\forall y \, \forall x \; Parent(x,y) \rightarrow Child(y,x)$

Translation exercise

Suppose:

- Variables x,y denote people
- L(x,y) denotes "x loves y".

Translate:

- Everybody loves Raymond. $\forall x\, L(x,Raymond)$
- Everybody loves somebody. $\forall x \exists y\, L(x,y)$
- There is somebody whom everybody loves. $\exists y \forall x\, L(x,y)$
- There is somebody who Raymond doesn't love.
  $\exists y \neg L(Raymond,y)$
- There is somebody whom no one loves.

  $\exists y\, \forall x\, \neg L(x,y)$

Negation of quantifiers

English statement:

- Nothing is perfect.
- **Translation:** $\neg \exists x$ Perfect(x)

Another way to express the same meaning:

- **Everything is imperfect.**
- **Translation:** $\forall x \neg$ Perfect(x)

Conclusion: $\neg \exists x\ P(x)$ is equivalent to $\forall x \neg P(x)$

Negation of quantifiers

English statement:

- It is not the case that all dogs are fleabags.
- **Translation:** $\neg \forall x \; Dog(x) \rightarrow Fleabag(x)$

Another way to express the same meaning:

- There is a dog that is not a fleabag.
- **Translation:** $\exists x \; Dog(x) \wedge \neg Fleabag(x)$

- Logically equivalent to:
  - $\exists x \neg (Dog(x) \rightarrow Fleabag(x))$

Conclusion: $\neg \forall x \; P(x)$ is equivalent to $\exists x \neg P(x)$

Negation of quantified statements
(aka DeMorgan Laws for quantifiers)

Negation	Equivalent
$\neg\exists x\ P(x)$	$\forall x\ \neg P(x)$
$\neg\forall x\ P(x)$	$\exists x\ \neg P(x)$

Theorems and proofs

- The truth value of some statements about the world is obvious and is easy to assign
- The truth of other statements may not be obvious, …

  …. But it may still follow (be derived) from known facts about the world

To show the truth value of such a statement following from other statements we need to provide **a correct supporting argument**

- **a proof**

Problem:

- It is easy to make a mistake and argue the support incorrectly.

Important questions:

- When is the argument correct?
- How to construct a correct argument, what method to use?

Theorems and proofs

- **Theorem:** a statement that can be shown to be true.
  - **Typically the theorem looks like this:**

$$\underbrace{(p1 \wedge p2 \wedge p3 \wedge \ldots \wedge pn)}_{\text{Premises (hypotheses)}} \rightarrow \underbrace{q}_{\text{conclusion}}$$

- **Example:**

  Fermat's Little theorem:
  - If p is a prime and a is an integer not divisible by p, then: $a^{p-1} \equiv 1 \mod p$

Theorems and proofs

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- **Example:**

  Fermat's Little theorem:
  - If p is a prime and a is an integer not divisible by p, then: $a^{p-1} \equiv 1 \bmod p$

Premises (hypotheses): If p is a prime and a is an integer not divisible by p

conclusion: $a^{p-1} \equiv 1 \bmod p$

Formal proofs

Proof:

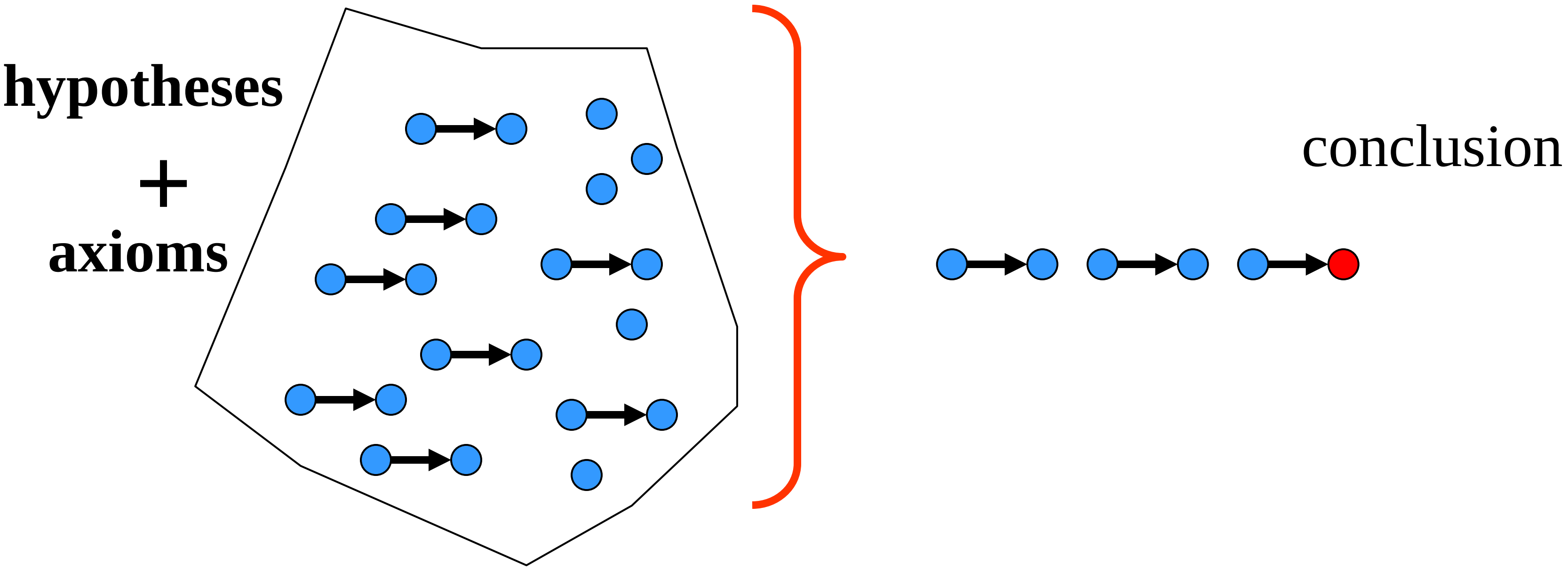
- Provides an argument supporting the validity of the statement
- Proof of the theorem:
  - shows that the conclusion follows from premises
  - may use:
    - Premises
    - Axioms
    - Results of other theorems

Formal proofs:

- steps of the proofs **follow logically** from the set of premises and axioms

Formal proofs

- **Formal proofs:**
  - show that steps of the proofs follow logically from the set of hypotheses and axioms



In this class we assume formal proofs in the propositional logic

Rules of inference

Rules of inference: logically valid inference patterns

Example;

- **Modus Ponens**, or the Law of Detachment
- Rule of inference

$$\begin{array}{l} p \\ \underline{p \rightarrow q} \\ \therefore\ q \end{array}$$

- Given p is true and the implication $p \rightarrow q$ is true then q is true.

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p	q	$p \rightarrow q$
False	*False*	*True*
False	*True*	*True*
True	*False*	*False*
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- Given p is true and the implication $p \rightarrow q$ is true then q is true.
- **Tautology Form:** $(p \wedge (p \rightarrow q)) \rightarrow q$