CS 441 Discrete Mathematics for CS
Lecture 4

Predicate logic

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Propositional logic: review

- **Propositional logic:** a formal language for making logical inferences
- **A proposition is** a statement that is either true or false.
- **A compound proposition** can be created from other propositions using logical connectives
- **The truth of a compound proposition** is defined by truth values of elementary propositions and the meaning of connectives.
- **The truth table for a compound proposition:** table with entries (rows) for all possible combinations of truth values of elementary propositions.

Tautology and Contradiction

Definitions:

- A compound proposition that is always true for all possible truth values of the propositions is called a **tautology**.
- A compound proposition that is always false is called a **contradiction**.
- A proposition that is neither a tautology nor contradiction is called a **contingency**.

Example: $p \wedge \neg p$ is a **contradiction.**

p	¬p	p ∧ ¬p
T	F	F
F	T	F

Equivalence

- Some propositions may be equivalent. Their truth values in the truth table are the same.
- Example: $\mathbf{p \rightarrow q}$ is equivalent to $\mathbf{\neg q \rightarrow \neg p}$ **(contrapositive)**

p	q	p → q	¬q → ¬p
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

- **Equivalent statements** are important for logical reasoning since they can be substituted and can help us to make a logical argument.

Important logical equivalences

- **Identity**
  - $p \wedge T \Leftrightarrow p$
  - $p \vee F \Leftrightarrow p$
- **Domination**
  - $p \vee T \Leftrightarrow T$
  - $p \wedge F \Leftrightarrow F$
- **Idempotent**
  - $p \vee p \Leftrightarrow p$
  - $p \wedge p \Leftrightarrow p$

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Showing logical equivalence

Example: Show $(p \wedge q) \rightarrow p$ is a tautology

In other words $((p \wedge q) \rightarrow p \Leftrightarrow T)$

Proof via truth table:

p	q	$p \wedge q$	$(p \wedge q) \rightarrow p$
T	T	T	**T**
T	F	F	**T**
F	T	F	**T**
F	F	F	**T**

Showing logical equivalences

- **Equivalences can be used in proofs.** A proposition or its part can be transformed using equivalences and some conclusion can be reached.

Example: Show that $(p \wedge q) \rightarrow p$ is a tautology.

- Proof: (we must show $(p \wedge q) \rightarrow p \iff T$)

$(p \wedge q) \rightarrow p$	$\iff \neg(p \wedge q) \vee p$	Useful
	$\iff [\neg p \vee \neg q] \vee p$	DeMorgan
	$\iff [\neg q \vee \neg p] \vee p$	Commutative
	$\iff \neg q \vee [\neg p \vee p]$	Associative
	$\iff \neg q \vee [T]$	Useful
	$\iff T$	Domination

Propositional logic

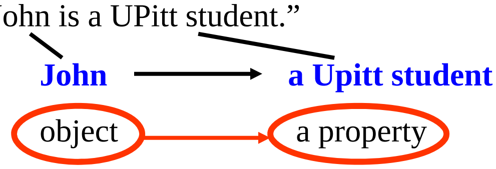
- **Definition:**
  - **A proposition is a statement that is either true or false.**
- **Examples:**
  - Pitt is located in the Oakland section of Pittsburgh.
  - 5 + 2 = 8.
  - It is raining today
  - 2 is a prime number
  - If (you do not drive over 65 mph) then (you will not get a speeding ticket).
- **Not a proposition:**
  - How are you?
  - x + 5 = 3

Limitations of the propositional logic

Propositional logic: the world is described in terms of elementary propositions and their logical combinations

Elementary statements:

- **Typically refer to objects, their properties and relations. But these are not explicitly represented** in the propositional logic
  - **Example:**
    - "John is a UPitt student."



- Objects and properties are hidden in the statement, it is not possible to reason about them

Limitations of the propositional logic

(1) Statements that must be repeated for many objects

– In propositional logic these must be exhaustively enumerated

- **Example:**

– If John is a CS UPitt graduate then John has passed cs441

Translation:

– John is a CS UPitt graduate → John has passed cs441

Similar statements can be written for other Upitt graduates:

– Ann is a CS Upitt graduate → Ann has passed cs441

– Ken is a CS Upitt graduate → Ken has passed cs441

– …

- **What is a more natural solution to express the above knowledge?**

Limitations of the propositional logic

(1) Statements that must be repeated for many objects

- **Example:**
  - If John is a CS UPitt graduate then John has passed cs441

  Translation:
  - John is a CS UPitt graduate → John has passed cs441

  Similar statements can be written for other Upitt graduates:
  - Ann is a CS Upitt graduate → Ann has passed cs441
  - Ken is a CS Upitt graduate → Ken has passed cs441
  - **…**
- **Solution:** make statements with **variables**
  - If x is a CS Upitt graduate then x has passed cs441
  - x is a CS UPitt graduate → x has passed cs441

Limitations of the propositional logic

(2) Statements that define the property of the group of objects

- **Example:**
  - All new cars must be registered.
  - Some of the CS graduates graduate with honors.
- **Solution:** make statements with **quantifiers**
  - **Universal quantifier** –the property is satisfied by all members of the group
  - **Existential quantifier** – at least one member of the group satisfy the property

Predicate logic

Remedies the limitations of the propositional logic

- Explicitly models objects and their properties
- Allows to make statements with variables and quantify them

Basic building blocks of the predicate logic:

- **Constant** –models a specific object

  Examples: "John", "France", "7"
- **Variable** – represents object of specific type **(defined by the *universe of discourse*)**

  Examples: x, y

  (universe of discourse can be people, students, numbers)
- **Predicate -** over one, two or many variables or constants.
  - Represents properties or relations among objects

  Examples: Red(car23), student(x), married(John,Ann)

Predicates

Predicates represent properties or relations among objects

A predicate P(x) assigns a value **true or false** to each x depending on whether the property holds or not for x.

- The assignment is best viewed as a big table with the variable x substituted for objects from ***the universe of discourse***

Example:

- Assume **Student(x)** where the universe of discourse are people
- Student(John) …. T (if John is a student)
- Student(Ann) …. T (if Ann is a student)
- Student(Jane) ….. F (if Jane is not a student)
- …

Predicates

Assume a predicate P(x) that represents the statement:

- **x is a prime number**

What are the truth values of:

- P(2) T
- P(3) T
- P(4) F
- P(5) T
- P(6) F
- P(7) T

Predicates

Assume a predicate P(x) that represents the statement:

- **x is a prime number**

What are the truth values of:

- P(2) T
- P(3) T
- P(4) F
- P(5) T
- P(6) F
- P(7) T

All statements P(2), P(3), P(4), P(5), P(6), P(7) are propositions

Predicates

Assume a predicate P(x) that represents the statement:
- **x is a prime number**

What are the truth values of:
- P(2) T
- P(3) T
- P(4) F
- P(5) T
- P(6) F
- P(7) T

Is P(x) a proposition? No. Many possible substitutions are possible.

Predicates

- Predicates can have **more arguments** which represent the **relations between objects**

Example:
- Let Q(x,y) denote 'x+5 >y'
  - Is Q(x,y) a proposition? **No!**
  - Is Q(3,7) a proposition? **Yes.** It is true.
  - What is the truth value of
  - Q(3,7) T
  - Q(1,6) F
  - Q(2,2) T
  - Is Q(3,y) a proposition? **No!** We cannot say if it is true or false.

Compound statements in predicate logic

Compound statements are obtained via logical connectives

Examples:

Student(Ann) $\wedge$ Student(Jane)

- **Translation:** "Both Ann and Jane are students"
- **Proposition:** yes.

Country(Sienna) $\vee$ River(Sienna)

- **Translation:** "Sienna is a country or a river"
- **Proposition:** yes.

CS-major(x) → Student(x)

- **Translation:** "if x is a CS-major then x is a student"
- **Proposition:** no.

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Predicates

Important:

- statement P(x) is **not a proposition** since there are more objects it can be applied to

This is the same as in propositional logic …

… But the difference is:

- predicate logic allows us to explicitly manipulate and substitute for the objects
- Predicate logic permits quantified sentences where variables are substituted for statements about the group of objects

Quantified statements

Predicate logic lets us to make statements about groups of objects

- To do this we use special quantified expressions

Two types of quantified statements:

- **universal**

  Example: ‘ all CS Upitt graduates have to pass cs441”
  - the statement is true for all graduates
- **existential**

  Example: ‘Some CS Upitt students graduate with honor.’
  - the statement is true for some people

Universal quantifier

Defn: The universal quantification of P(x) is the proposition: "*P(x) is true for all values of x in the domain of discourse*." The notation **∀x P(x)** denotes the universal quantification of P(x), and is expressed as **for every x, P(x).**

Example:

- Let P(x) denote x > x - 1.
- What is the truth value of ∀x P(x)?
- Assume the universe of discourse of x are all real numbers.

Universal quantifier

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Example:

- Let P(x) denote x > x - 1.
- What is the truth value of ∀x P(x)?
- Assume the universe of discourse of x is all real numbers.
- **Answer:** Since every number x is greater than itself minus 1. Therefore, **∀x P(x) is true.**

Universal quantifier

Quantification converts a propositional function into **a proposition** by binding a variable to a set of values from the universe of discourse.

Example:

- Let P(x) denote x > x - 1.
- Is P(x) a proposition? **No.** Many possible substitutions.
- Is ∀x P(x) a proposition? **Yes.** True if for all x from the universe of discourse P(x) is true.

Universally quantified statements

Predicate logic lets us make statements about groups of objects

Universally quantified statement

- CS-major(x) → Student(x)
  - **Translation:** "if x is a CS-major then x is a student"
  - **Proposition:** **no.**

- ∀x CS-major(x) → Student(x)
  - **Translation:** "(For all people it holds that) if a person is a CS-major then she is a student."
  - **Proposition:** **yes.**

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Existential quantifier

Definition: The **existential quantification** of P(x) is the proposition "*There exists an element in the domain (universe) of discourse such that P(x) is true*." The notation **∃x P(x)** denotes the existential quantification of P(x), and is expressed as **there is an x such that P(x) is true**.

Example 1:

- Let T(x) denote x > 5 and x is from Real numbers.
- What is the truth value of ∃x T(x)?
- **Answer:**
- Since 10 > 5 is true. Therefore, it is **true that ∃x T(x).**

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Existential quantifier

Definition: The **existential quantification** of P(x) is the proposition "*There exists an element in the domain (universe) of discourse such that P(x) is true*." The notation **∃x P(x)** denotes the existential quantification of P(x), and is expressed as **there is an x such that P(x) is true**.

Example 2:

- Let Q(x) denote x = x + 2 where x is real numbers
- What is the truth value of ∃x Q(x)?
- **Answer:** Since no real number is 2 larger than itself, the truth value of **∃x Q(x) is false**.