CS 441 Discrete Mathematics for CS
Lecture 3

Propositional logic

Milos Hauskrecht
milos@cs.pitt.edu
5329 Sennott Square

Propositional logic: review

- **Propositional logic:** a formal language for making logical inferences
- **A proposition is** a statement that is either true or false.
- **A compound proposition** can be created from other propositions using logical connectives
- **The truth of a compound proposition** is defined by truth values of elementary propositions and the meaning of connectives.
- **The truth table for a compound proposition:** table with entries (rows) for all possible combinations of truth values of elementary propositions.

Compound propositions

- Let p: 2 is a prime ….. **T**
  q: 6 is a prime ….. **F**
- Determine **the truth value** of the following statements:

$\neg p$:

$p \wedge q$:

$p \wedge \neg q$:

$p \vee q$:

$p \oplus q$:

$p \rightarrow q$:

$q \rightarrow p$:

Compound propositions

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- Determine **the truth value** of the following statements:

$\neg p$: **F**

$p \wedge q$: **F**

$p \wedge \neg q$: **T**

$p \vee q$: **T**

$p \oplus q$: **T**

$p \rightarrow q$: **F**

$q \rightarrow p$: **T**

Constructing the truth table

- **Example:** Construct the truth table for
  $(p \rightarrow q) \wedge (\neg p \leftrightarrow q)$

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Rows: all possible combinations of values for elementary propositions: 2^n values

p	q	¬p			(p→q)∧ (¬p↔q)
T	T				
T	F				
F	T				
F	F				

Constructing the truth table

- Example: Construct the truth table for
  $(p \rightarrow q) \wedge (\neg p \leftrightarrow q)$

Typically the target (unknown) compound proposition and its values

p	q	¬p	p → q	¬p ↔ q	(p→q)∧ (¬p↔q)
T	T				
T	F				
F	T				
F	F				

Auxiliary compound propositions and their values

Constructing the truth table

- Examples: Construct a truth table for
  $(p \rightarrow q) \wedge (\neg p \leftrightarrow q)$

p	q	¬p	p → q	¬p ↔ q	(p→q)∧ (¬p↔q)
T	T	F	T	F	F
T	F	F	F	T	F
F	T	T	T	T	T
F	F	T	T	F	F

Computer representation of True and False

We need to encode two values True and False:

- **Computers represents data and programs using 0s and 1s**
- Logical truth values – True and False
- A bit is sufficient to represent two possible values:
  - 0 (False) or 1(True)

- A variable that takes on values 0 or 1 is called a **Boolean variable**.

- Definition: A **bit string** is a sequence of zero or more bits. The **length** of this string is the number of bits in the string.

Bitwise operations

- T and F replaced with 1 and 0

p	q	$p \vee q$	$p \wedge q$
1	1	1	1
1	0	1	0
0	1	1	0
0	0	0	0

p	$\neg p$
1	0
0	1

Bitwise operations

- **Examples:**

$$\begin{array}{r}1011\ 0011\\ \vee\ \underline{0110\ 1010}\end{array}\qquad \begin{array}{r}1011\ 0011\\ \wedge\ \underline{0110\ 1010}\end{array}\qquad \begin{array}{r}1011\ 0011\\ \oplus\ \underline{0110\ 1010}\end{array}$$

Bitwise operations

- **Examples:**

$$\begin{array}{r}1011\ 0011\\ \vee\ \underline{0110\ 1010}\\ 1111\ 1011\end{array}\qquad \begin{array}{r}1011\ 0011\\ \wedge\ \underline{0110\ 1010}\\ 0010\ 0010\end{array}\qquad \begin{array}{r}1011\ 0011\\ \oplus\ \underline{0110\ 1010}\\ 1101\ 1001\end{array}$$

Applications of propositional logic

- **Translation of English sentences**
- **Artificial intelligence:**
  - Representation of knowledge about the world
  - Inferences about the knowledge
- **Logic circuits**

Translation

If you are older than 13 or you are with your parents then you can attend a PG-13 movie.

Parse:

- **If (** you are older than 13 **or** you are with your parents **) then (**you can attend a PG-13 movie**)**
  - A= you are older than 13
  - B= you are with your parents
  - C=you can attend a PG-13 movie
- **Translation:** $A \vee B \rightarrow C$

- **But why do we want to do it this way?**

Translation

- **General rule for translation.**
- Look for patterns corresponding to logical connectives in the sentence and use them to define elementary propositions.

- **Example:**

You can have free coffee if you are senior citizen and it is a Tuesday

Step 1 find logical connectives

Translation

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- **Example:**

You can have free coffee **if** you are senior citizen **and** it is a Tuesday

Step 2 break the sentence into elementary propositions

Translation

- **General rule for translation.**
- Look for patterns corresponding to logical connectives in the sentence and use them to define elementary propositions.

- **Example:**

You can have free coffee **if** you are senior citizen **and** it is a Tuesday

(a: You can have free coffee; b: you are senior citizen; c: it is a Tuesday)

Step 2 break the sentence into elementary propositions

Translation

- **General rule for translation .**
- Look for patterns corresponding to logical connectives in the sentence and use them to define elementary propositions.

- **Example:**

You can have free coffee if you are senior citizen and it is a Tuesday

a **b** **c**

Step 3 rewrite the sentence in propositional logic

$$\mathbf{b} \land \mathbf{c} \rightarrow \mathbf{a}$$

Translation

- Assume two elementary statements:
  - **p: you drive over 65 mph** ; **q: you get a speeding ticket**
- **Translate each of these sentences to logic**
  - you do not drive over 65 mph. $(\neg p)$
  - you drive over 65 mph, but you don't get a speeding ticket. $(p \land \neg q)$
  - you will get a speeding ticket if you drive over 65 mph. $(p \rightarrow q)$
  - if you do not drive over 65 mph then you will not get a speeding ticket.$(\neg p \rightarrow \neg q)$
  - driving over 65 mph is sufficient for getting a speeding ticket. $(p \rightarrow q)$
  - you get a speeding ticket, but you do not drive over 65 mph. $\mathbf{(q \land \neg p)}$

Applications: logical inferences

Artificial Intelligence:

- Studies the design of programs that behave/act intelligently
- Many AI problems require the ability to represent and reason with the knowledge about the domain of interest

- **Expert systems**:
  - Represent the knowledge about the world or domain of interest (e.g. medicine) using some kind of logic
  - Support inferences consistent with the logic

Example: MYCIN

- MYCIN: an expert system for diagnosis of bacterial infections
- **Knowledge (represented in propositional logic) consists of:**
  - Facts about a specific patient case
  - Rules describing relations between entities in the bacterial infection domain

If	1. The stain of the organism is gram-positive, and 2. The morphology of the organism is coccus, and 3. The growth conformation of the organism is chains
Then	the identity of the organism is streptococcus

- **Inferences**:
  - manipulates the facts and known relations to answer diagnostic queries (consistent with findings and rules)

Tautology and Contradiction

Definitions:

- A compound proposition that is always true for all possible truth values of the propositions is called a **tautology**.
- A compound proposition that is always false is called a **contradiction**.
- A proposition that is neither a tautology nor contradiction is called a **contingency**.
-

Example: $p \lor \neg p$ is a **tautology.**

p	¬p	p ∨ ¬p
T	F	T
F	T	T

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Example: $p \land \neg p$ is a **contradiction.**

p	¬p	p ∧ ¬p
T	F	F
F	T	F

Equivalence

- Some propositions may be equivalent. Their truth values in the truth table are the same.
- Example: **p → q** is equivalent to **¬q → ¬p (contrapositive)**

p	q	p → q	¬q → ¬p
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

- **Equivalent statements** are important for logical reasoning since they can be substituted and can help us to make a logical argument.

Logical equivalence

- **Definition**: The propositions p and q are called **logically equivalent** if p ↔ q is a tautology (alternately, if they have the same truth table). The notation **p <=> q** denotes p and q are logically equivalent.

Examples of equivalences:

- **DeMorgan's Laws:**
- 1) ¬(p ∨ q) <=> ¬p ∧ ¬q
- 2) ¬(p ∧ q) <=> ¬p ∨ ¬q

Equivalence

Example of important equivalences

- **DeMorgan's Laws:**
- 1) ¬(p ∨ q) <=> ¬p ∧ ¬q
- 2) ¬(p ∧ q) <=> ¬p ∨ ¬q

Use the truth table to prove that the two propositions are logically equivalent

p	q	¬p	¬q	¬(p ∨ q)	¬p ∧ ¬q
T	T	F	F	F	F
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

Equivalence

Example of important equivalences

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- 2) ¬(p ∧ q) <=> ¬p ∨ ¬q

Use the truth table to prove that the two propositions are logically equivalent

p	q	¬p	¬q	¬(p ∨ q)	¬p ∧ ¬q
T	T	F	F	**F**	**F**
T	F	F	T	**F**	**F**
F	T	T	F	**F**	**F**
F	F	T	T	**T**	**T**

Important logical equivalences

- **Identity**
  - $p \wedge T \iff p$
  - $p \vee F \iff p$
- **Domination**
  - $p \vee T \iff T$
  - $p \wedge F \iff F$
- **Idempotent**
  - $p \vee p \iff p$
  - $p \wedge p \iff p$

Important logical equivalences

- **Double negation**
  - $\neg(\neg p) \iff p$
- **Commutative**
  - $p \vee q \iff q \vee p$
  - $p \wedge q \iff q \wedge p$
- **Associative**
  - $(p \vee q) \vee r \iff p \vee (q \vee r)$
  - $(p \wedge q) \wedge r \iff p \wedge (q \wedge r)$

Important logical equivalences

- **Distributive**
  - $p \vee (q \wedge r) \iff (p \vee q) \wedge (p \vee r)$
  - $p \wedge (q \vee r) \iff (p \wedge q) \vee (p \wedge r)$

- **De Morgan**
  - $\neg(p \vee q) \iff \neg p \wedge \neg q$
  - $(p \wedge q) \iff \neg p \vee \neg q$

- **Other useful equivalences**
  - $p \vee \neg p \iff T$
  - $p \wedge \neg p \iff F$
  - **$p \rightarrow q \iff (\neg p \vee q)$**

Using logical equivalences

- **Equivalences can be used in proofs.** A proposition or its part can be transformed using equivalences and some conclusion can be reached.

Example: Show that $(p \wedge q) \rightarrow p$ is a tautology.

- Proof: (we must show $(p \wedge q) \rightarrow p \iff T$)

  $(p \wedge q) \rightarrow p \iff \neg(p \wedge q) \vee p$ Useful

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- Proof: (we must show $(p \wedge q) \rightarrow p$ <=> T)

$(p \wedge q) \rightarrow p \iff \neg(p \wedge q) \vee p$ Useful

$\iff [\neg p \vee \neg q] \vee p$ DeMorgan

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$(p \wedge q) \rightarrow p \iff \neg(p \wedge q) \vee p$ Useful

$\iff [\neg p \vee \neg q] \vee p$ DeMorgan

$\iff [\neg q \vee \neg p] \vee p$ Commutative

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- Proof: (we must show $(p \wedge q) \rightarrow p \;<=>\; T$)

$(p \wedge q) \rightarrow p \;<=>\; \neg(p \wedge q) \vee p$ — Useful

$<=>\; [\neg p \vee \neg q] \vee p$ — DeMorgan

$<=>\; [\neg q \vee \neg p] \vee p$ — Commutative

$<=>\; \neg q \vee [\neg p \vee p]$ — Associative

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- Proof: (we must show $(p \wedge q) \rightarrow p \;<=>\; T$)

$(p \wedge q) \rightarrow p \;<=>\; \neg(p \wedge q) \vee p$ — Useful

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$<=>\; [\neg q \vee \neg p] \vee p$ — Commutative

$<=>\; \neg q \vee [\neg p \vee p]$ — Associative

$<=>\; \neg q \vee [T]$ — Useful

Using logical equivalences

- **Equivalences can be used in proofs.** A proposition or its part can be transformed using equivalences and some conclusion can be reached.

Example: Show that $(p \wedge q) \rightarrow p$ is a tautology.

- Proof: (we must show $(p \wedge q) \rightarrow p$ <=> T)

$(p \wedge q) \rightarrow p$ <=>	$\neg(p \wedge q) \vee p$	Useful
<=>	$[\neg p \vee \neg q] \vee p$	DeMorgan
<=>	$[\neg q \vee \neg p] \vee p$	Commutative
<=>	$\neg q \vee [\neg p \vee p]$	Associative
<=>	$\neg q \vee [T]$	Useful
<=>	T	Domination

Using logical equivalences

- **Equivalences can be used in proofs.** A proposition or its part can be transformed using equivalences and some conclusion can be reached.

Example: Show $(p \wedge q) \rightarrow p$ is a tautology.

- Alternative proof:

p	q	$p \wedge q$	$(p \wedge q) \rightarrow p$
T	T	T	**T**
T	F	F	**T**
F	T	F	**T**
F	F	F	**T**

Using logical equivalences

- **Equivalences can be used in proofs.** A proposition or its part can be transformed using equivalences and some conclusion can be reached.

- **Proofs that rely on logical equivalences can replace truth table approach**
  - **Why?**
  - The truth table has 2^n rows, where n is the number of elementary propositions
  - If n is large building the truth table may become infeasible

Using logical equivalences

- **Equivalences can be used in proofs.** A proposition or its part can be transformed using equivalences and some conclusion can be reached.

- **Example 2:** Show $(p \rightarrow q) <=> (\neg q \rightarrow \neg p)$

  Proof:
- $(p \rightarrow q) <=> (\neg q \rightarrow \neg p)$
- $<=>$?

Using logical equivalences

- Equivalences can be used in proofs. A proposition or its part can be transformed using equivalences and some conclusion can be reached.

- **Example 2:** Show $(p \rightarrow q) \Leftrightarrow (\neg q \rightarrow \neg p)$

  Proof:

- $(p \rightarrow q) \Leftrightarrow (\neg q \rightarrow \neg p)$
- $\Leftrightarrow \neg(\neg q) \vee (\neg p)$ Useful
- $\Leftrightarrow q \vee (\neg p)$ Double negation
- $\Leftrightarrow \neg p \vee q$ Commutative
- $\Leftrightarrow p \rightarrow q$ Useful

  End of proof