CS 441 Discrete Mathematics for CS
Lecture 24

Relations V.

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Equivalence relation

Definition: A relation R on a set A is called an **equivalence relation** if it is **reflexive, symmetric and transitive.**

Equivalence relation

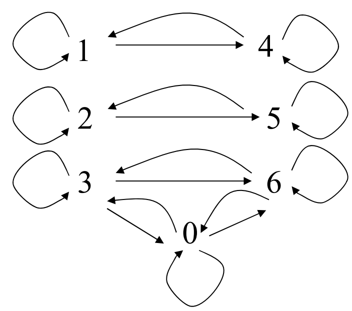
- **Relation R on A={0,1,2,3,4,5,6} has the following pairs:**

  (0,0) (0,3), (3,0), (0,6), (6,0)

  (3,3), (3,6) (6,3), (6,6) (1,1),(1,4), (4,1), (4,4)

  (2,2), (2,5), (5,2), (5,5)

- Is R reflexive? **Yes.**
- Is R symmetric? **Yes.**
- Is R transitive. **Yes.**



Then

- **R is an equivalence relation**.

Equivalence class

Definition: Let R be an equivalence relation on a set A. The set { x ∈ A | a R x} is called **the equivalence class of a,** denoted by $[a]_R$ or simply [a] when there is only one relation R. If b ∈ [a] then b is called **a representative of this equivalence class**.

Example:

- Assume R={(a,b) | a ≡ b mod 3} for A={0,1,2,3,4,5,6}
- Pick an element a =0.
- $[0]_R = \{0,3,6\}$
- Element 1: $[1]_R = \{1,4\}$
- Element 2: $[2]_R = \{2,5\}$
- Element 3: $[3]_R = \{0,3,6\} = [0]_R = [6]_R$
- Element 4: $[4]_R = \{1,4\} = [1]_R$
- Element 5: $[5]_R = \{2,5\} = [2]_R$

Equivalence class

Example:

- Assume R={(a,b) | a ≡ b mod 3} for A={0,1,2,3,4,5,6}

Three different equivalence classes all together:

- $[0]_R = [3]_R = [6]_R = \{0,3,6\}$
- $[1]_R = [4]_R = \{1,4\}$
- $[2]_R = [5]_R = \{2,5\}$

Partition of a set S

Definition: Let S be a set. A collection of nonempty subsets of S $A_1, A_2, \ldots, A_k$ is called **a partition of S** if:

- $A_i \cap A_j = \emptyset,\ i \neq j$ and $S = \bigcup_{i=1}^{k} A_i$

Example: Let S={1,2,3,4,5,6} and

- $A_1 = \{0,3,6\}$ $A_2 = \{1,4\}$ $A_3 = \{2,5\}$
- Is A_1, $A_{2,,}$ A_3 a partition of S ? **Yes.**

- Give a partition of S ?
- {0,2,4,6} {1,3,5}
- {0} {1,2} {3,4,5} {6}

Partial orderings

Definition: A relation *R* on a set S is called a *partial ordering*, or *partial order*, if it **is reflexive, antisymmetric, and transitive**. A set together with a partial ordering *R* is called a *partially ordered set*, or *poset*, and is denoted by (*S*, *R*). Members of *S* are called *elements* of the poset.

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Example: Assume R denotes the "greater than or equal" relation ($\geq$) on the set S={1,2,3,4,5}.

- Is the relation reflexive? Yes
- Is it antisymmetric? Yes
- Is it transitive? Yes
- **Conclusion:** R is a partial ordering.

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Comparability

Definition 1: The elements *a* and *b* of a poset (S,$\preccurlyeq$) are *comparable* if either $a \preccurlyeq b$ or $b \preccurlyeq a$. When *a* and *b* are elements of *S* so that neither $a \preccurlyeq b$ nor $b \preccurlyeq a$ holds, then *a* and *b* are called ***in**comparable*.

Definition 2: If (S,$\preccurlyeq$) is a poset and every two elements of *S* are comparable, *S* is called a *totally ordered* or *linearly ordered set*, and $\preccurlyeq$ is called a *total order* or a *linear order*. A totally ordered set is also called a *chain.*

Definition 3: (S,$\preccurlyeq$) is a well-ordered set if it is a poset such that $\preccurlyeq$ is a total ordering and every nonempty subset of *S* has **a least element**.

Lexicographical ordering

Definition: Given two posets $(A_1, \preccurlyeq_1)$ and $(A_2, \preccurlyeq_2)$, the *lexicographic ordering* on $A_1 \times A_2$ is defined by specifying that (a_1, a_2) is less than (b_1, b_2), that is,

$$(a_1, a_2) \prec (b_1, b_2),$$

either if $a_1 \prec_1 b_1$ or if $a_1 = b_1$ and $a_2 \prec_2 b_2$.

The definition can be extended to a lexicographic ordering on strings

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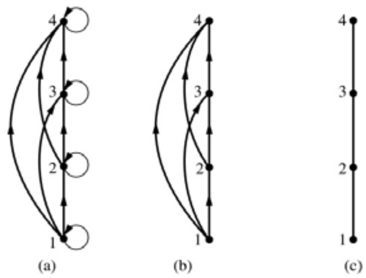
Example: Consider strings of lowercase English letters. A lexicographic ordering can be defined using the ordering of the letters in the alphabet. This is the same ordering as that used in dictionaries.

- *discreet* $\prec$ *discrete*, because these strings differ in the seventh position and $e \prec t$.
- *discreet* $\prec$ *discreetness*, because the first eight letters agree, but the second string is longer.

Hasse diagram

Definition: A *Hasse diagram* is a visual representation of a partial ordering that leaves out edges that must be present because of the reflexive and transitive properties.



(a) A partial ordering. The loops due to the reflexive property are

(b) The edges that must be present due to the transitive property are deleted in

(c) The Hasse diagram for the partial ordering (a).

Procedure for constructing Hasse diagram

- To represent a finite poset $(S, \preccurlyeq)$ using a Hasse diagram, start with the directed graph of the relation:
  - Remove the loops (a, a) present at every vertex due to the reflexive property.
  - Remove all edges (x, y) for which there is an element $z \in S$ such that $x \prec z$ and $z \prec y$. These are the edges that must be present due to the transitive property.
  - Arrange each edge so that its initial vertex is below the terminal vertex. Remove all the arrows, because all edges point upwards toward their terminal vertex.

CS 441 Discrete Mathematics for CS
Lecture 24b

Graphs
(chapter 10)

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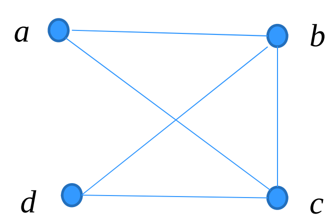
Definition of a graph

- **Definition:** A *graph* $G = (V, E)$ consists of a nonempty set V of *vertices* (or *nodes*) and a set E of *edges.* Each edge has either one or two vertices associated with it, called its *endpoints*. An edge is said to *connect* its endpoints.
- **Example:**



Terminology

- In a *simple graph* each edge connects two different vertices and no two edges connect the same pair of vertices.
- *Multigraphs* may have multiple edges connecting the same two vertices. When *m* different edges connect the vertices *u* and *v*, we say that {*u*,*v*} is an edge of *multiplicity m*.
- An edge that connects a vertex to itself is called a *loop*.
- A *pseudograph* may include loops, as well as multiple edges connecting the same pair of vertices.

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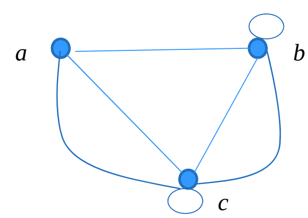


Directed graph

Definition: An *directed graph* (or *digraph*) $G = (V, E)$ consists of a nonempty set V of *vertices* (or *nodes*) and a set E of *directed edges* (or *arcs*). Each edge is associated with an ordered pair of vertices. The directed edge associated with the ordered pair (u,v) is said to *start at u* and *end at v*.

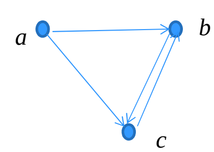
Remark:

– Graphs where the end points of an edge are not ordered are said to be ***undirected graphs.***

Directed graph

- A *simple directed graph* has no loops and no multiple edges.

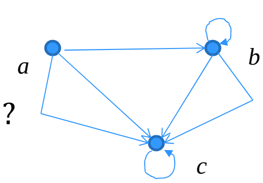
Example:



- A *directed multigraph* may have multiple directed edges. When there are m directed edges from the vertex u to the vertex v, we say that (u,v) is an edge of *multiplicity* m.

Example:

- multiplicity of (a,b) is ?
- and the multiplicity of (b,c) is ?

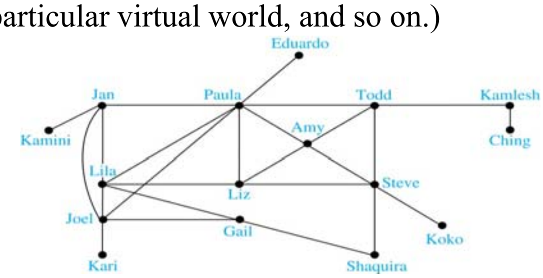


Graphs

- **Graphs and graph theory can be used to model:**
  - Computer networks
  - Social networks
  - Communications networks
  - Information networks
  - Software design
  - Transportation networks
  - Biological networks

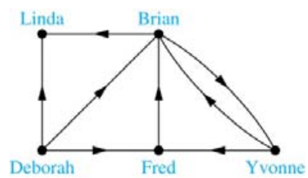
Graph models

- Graphs can be used to model social structures based on different kinds of relationships between people or groups.
- *Social network*, vertices represent individuals or organizations and edges represent relationships between them.
- Useful graph models of social networks include:
  - *friendship graphs* - undirected graphs where two people are connected if they are friends (in the real world, on Facebook, or in a particular virtual world, and so on.)



Graph models

- Useful graph models of social networks include:
  - *influence graphs* - directed graphs where there is an edge from one person to another if the first person can influence the second person



  - *collaboration graphs* - undirected graphs where two people are connected if they collaborate in a specific way

Collaboration graphs

- The *Hollywood graph* models the collaboration of actors in films.
  - We represent actors by vertices and we connect two vertices if the actors they represent have appeared in the same movie.
  - Kevin Bacon numbers.
- An *academic collaboration graph* models the collaboration of researchers who have jointly written a paper in a particular subject.
  - We represent researchers in a particular academic discipline using vertices.
  - We connect the vertices representing two researchers in this discipline if they are coauthors of a paper.
  - *Erdős number*

Information graphs

- Graphs can be used to model different types of networks that link different types of information.
- In a ***web graph***, web pages are represented by vertices and links are represented by directed edges.
  - A web graph models the web at a particular time.
- In a ***citation network***:
  - Research papers in a particular discipline are represented by vertices.
  - When a paper cites a second paper as a reference, there is an edge from the vertex representing this paper to the vertex representing the second paper.

Transportation graphs

- Graph models are extensively used in the study of transportation networks.
- **Airline networks** modeled using directed multigraphs:
  - airports are represented by vertices
  - each flight is represented by a directed edge from the vertex representing the departure airport to the vertex representing the destination airport
- **Road networks** can be modeled using graphs where
  - vertices represent intersections and edges represent roads.
  - undirected edges represent two-way roads and directed edges represent one-way roads.

Transportation graphs

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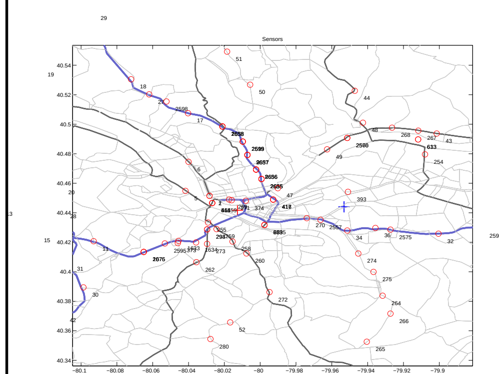
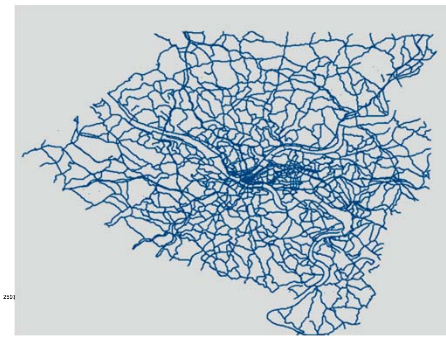




Graphs: basics

Basic types of graphs:

- Undirected graphs
- Directed graphs

Undirected graphs

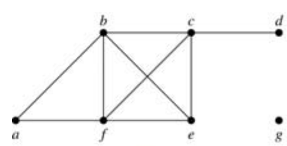
Definition 1. Two vertices u, v in an undirected graph G are called ***adjacent* (or *neighbors*)** in G if there is an edge e between u and v. Such an edge e is called *incident with* the vertices u and v and e is said to *connect* u and v.

Definition 2. The set of all neighbors of a vertex v of $G = (V, E)$, denoted by $N(v)$, is called **the *neighborhood* of v**. If A is a subset of V, we denote by $N(A)$ the set of all vertices in G that are adjacent to at least one vertex in A. So,

Definition 3. The ***degree of a vertex in a undirected graph*** is the number of edges incident with it, except that a loop at a vertex contributes two to the degree of that vertex. The degree of the vertex v is denoted by $\deg(v)$.

Undirected graphs

Example: What are the degrees and neighborhoods of the vertices in the graphs G?



G

Solution:

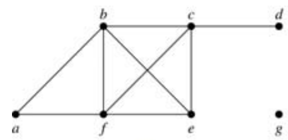
G: $\deg(a) = 2$, $\deg(b) = \deg(c) = \deg(f) = 4$, $\deg(d) = 1$, $\deg(e) = 3$, $\deg(g) = 0$.

$N(a) = ?$, $N(b) = ?$, $N(c) = ?$

$N(d) = ?$, $N(e) = ?$, $N(f) = ?$, $N(g) = ?$.

Undirected graphs

Example: What are the degrees and neighborhoods of the vertices in the graphs *G*?



G

Solution:

G: deg(a) = 2, deg(b) = deg(c) = deg(f) = 4, deg(d) = 1,

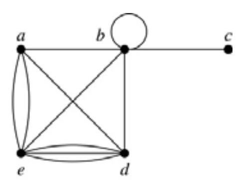
deg(e) = 3, deg(g) = 0.

$N(a) = \{b, f\}$, $N(b) = \{a, c, e, f\}$, $N(c) = \{b, d, e, f\}$,

$N(d) = \{c\}$, $N(e) = \{b, c, f\}$, $N(f) = \{a, b, c, e\}$, $N(g) = \varnothing$.

Undirected graphs

Example: What are the degrees and neighborhoods of the vertices in the graphs *H*?



H

Solution:

H: deg(a) = 4, deg(b) = deg(e) = 6, deg(c) = 1, deg(d) = 5.

$N(a) = \{b, d, e\}$, $N(b) = \{a, b, c, d, e\}$, $N(c) = \{b\}$,

$N(d) = \{a, b, e\}$, $N(e) = \{a, b, d\}$.

Undirected graphs

Theorem 1 (*Handshaking Theorem*): If $G = (V,E)$ is an undirected graph with m edges, then

$$2m = \sum_{v \in V} \deg(v)$$

Proof:

Each edge contributes twice to the degree count of all vertices. Hence, both the left-hand and right-hand sides of this equation equal twice the number of edges.

Think about the graph where vertices represent the people at a party and an edge connects two people who have shaken hands.

Undirected graphs

Theorem 2: An undirected graph has an even number of vertices of odd degree.

***Proof*:** Let V_1 be the vertices of even degree and V_2 be the vertices of odd degree in an undirected graph $G = (V, E)$ with m edges. Then

$$2m = \sum_{v \in V} \deg(v) = \sum_{v \in V_1} \deg(v) + \sum_{v \in V_2} \deg(v).$$

must be even since $\deg(v)$ is even for each $v \in V_1$

This sum must be even because $2m$ is even and the sum of the degrees of the vertices of even degrees is also even. Because this is the sum of the degrees of all vertices of odd degree in the graph, there must be an even number of such vertices.