## CS 441 Discrete Mathematics for CS
## Lecture 23

# Relations IV

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## Closures on relations

- **Relations can have different properties:**
  - **reflexive,**
  - **symmetric**
  - **transitive**

- Because of that we can have:
  - **symmetric,**
  - **reflexive and**
  - **transitive**

  **closures.**

# Closures

**Definition:** Let R be a relation on a set A. A relation S on A with property P is called **the closure of R with respect to P** if S is a subset of every relation Q ($S \subseteq Q$) with property P that contains R ($R \subseteq Q$).



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**Example (a symmetric closure):**

- Assume R={(1,2),(1,3), (2,2)} on A={1,2,3}.
- What is the symmetric closure S of R?
- S = {(1,2),(1,3), (2,2)} ∪ **{(**2,1), (3,1)}
  = {(1,2),(1,3), (2,2),(2,1), (3,1)}

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**Example (transitive closure):**

- Assume R={(1,2), (2,2), (2,3)} on A={1,2,3}.
- **Is R transitive? No.**
- **How to make it transitive?**
- **S = ?**



## Closures

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**Example (transitive closure):**

- Assume R={(1,2), (2,2), (2,3)} on A={1,2,3}.
- **Is R transitive? No.**
- **How to make it transitive?**
- **S = {(1,2), (2,2), (2,3)} $\cup$ {(1,3)}**
  **= {(1,2), (2,2), (2,3),(1,3)}**
- S is the transitive closure of R

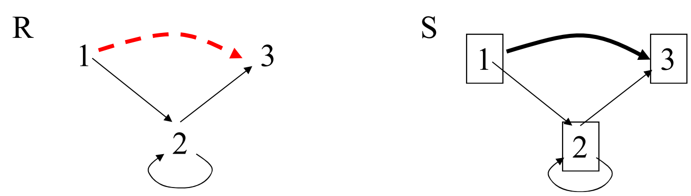
# Transitive closure

We can represent the relation on the graph. Finding a transitive closure corresponds to finding all pairs of elements that are connected with a directed path (or digraph).

**Example:**

Assume R={(1,2), (2,2), (2,3)} on A={1,2,3}.

Transitive closure S = {(1,2), (2,2), (2,3),(1,3)}.



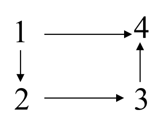



# Connectivity relation

**Definition:** Let R be a relation on a set A. The **connectivity relation** R* consists of all pairs (a,b) such that there is a path (of any length, ie. 1 or 2 or 3 or ...) between a and b in R.

$$R^* = \bigcup_{k=1}^{\infty} R^k$$

**Example:**




- A = {1,2,3,4}
- R = {(1,2),(1,4),(2,3),(3,4)}
- $R^2$ = {(1,3),(2,4)}
- $R^3$ = {(1,4)}
- $R^4 = \varnothing$
- ...
- R* = {(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)}

## Transitivity closure and connectivity relation

**Theorem:** The transitive closure of a relation R **equals** the connectivity relation R*.

**Lemma:** Let A be a set with n elements, and R a relation on A. If there is a path from a to b, then there exists a path of length $<$ n in between (a,b). Consequently:

$$R^* = \bigcup_{k=1}^{n} R^k$$



## Equivalence relation

**Definition:** A relation R on a set A is called an **equivalence relation** if it is **reflexive, symmetric and transitive**.

**Example:** Let A = {0,1,2,3,4,5,6} and

- R= {(a,b)| a,b $\in$ A, a $\equiv$ b mod 3} (a is congruent to b modulo 3)

**Congruencies:**

- 0 mod 3 = 0 1 mod 3 = 1 2 mod 3 = 2 3 mod 3 = 0
- 4 mod 3 = 1 5 mod 3 = 2 6 mod 3 = 0

**Relation R has the following pairs:**

- (0,0) (0,3), (3,0), (0,6), (6,0)
- (3,3), (3,6) (6,3), (6,6) (1,1),(1,4), (4,1), (4,4)
- (2,2), (2,5), (5,2), (5,5)

## Equivalence relation

**Example:**

- Suppose that *R* is the relation on the set of strings of English letters such that *aRb* if and only if $l(a) = l(b)$, where $l(x)$ is the length of the string *x*. Is *R* an equivalence relation?
- **Is the relation reflexive?**

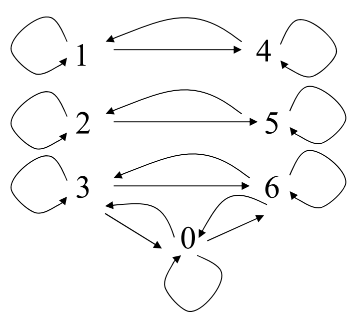


## Equivalence relation

- **Relation R on A={0,1,2,3,4,5,6} has the following pairs:**

(0,0) (0,3), (3,0), (0,6), (6,0)
(3,3), (3,6) (6,3), (6,6) (1,1),(1,4), (4,1), (4,4)
(2,2), (2,5), (5,2), (5,5)

- Is R reflexive? **Yes.**
- Is R symmetric? **Yes.**
- Is R transitive. **Yes.**




**Then**

- **R is an equivalence relation**.

## Equivalence relation

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- Suppose that *R* is the relation on the set of strings of English letters such that *aRb* if and only if $l(a) = l(b)$, where $l(x)$ is the length of the string *x*. Is *R* an equivalence relation?
- **Is the relation reflexive? Yes**.
  - Because $l(a) = l(a)$, it follows that *aRa* for all strings *a*.
- **Is the relation symmetric? Yes.**
  - Suppose that *aRb*. Since $l(a) = l(b)$, $l(b) = l(a)$ also holds and *bRa*.
- **Is the relation transitive? Yes.**
  - Suppose that a*R*b and *bRc*. Since $l(a) = l(b)$,and $l(b) = l(c)$, $l(a) = l(a)$ also holds and *aRc*.
- **Conclusion: *R* an equivalence relation**



## Equivalence class

**Definition:** Let R be an equivalence relation on a set A. The set $\{ x \in A \mid a R x\}$ is called **the equivalence class of a,** denoted by $[a]_R$ or simply [a] when there is only one relation R. If $b \in [a]$ then b is called **a representative of this equivalence class**.

**Example:**

- Assume $R=\{(a,b) \mid a \equiv b \bmod 3\}$ for $A=\{0,1,2,3,4,5,6\}$
- Pick an element a =0.
- $[0]_R = \{0,3,6\}$
- Element 1: $[1]_R= \{1,4\}$
- Element 2: $[2]_R= \{2,5\}$
- Element 3: $[3]_R= \{0,3,6\} = [0]_R = [6]_R$
- Element 4: $[4]_R= \{1,4\} = [1]_R$
- Element 5: $[5]_R= \{2,5\} = [2]_R$

## Equivalence class

**Example:**

- Assume R={(a,b) | a ≡ b mod 3} for A={0,1,2,3,4,5,6}

**Three different equivalence classes all together:**

- $[0]_R = [3]_R = [6]_R = \{0,3,6\}$
- $[1]_R = [4]_R = \{1,4\}$
- $[2]_R = [5]_R = \{2,5\}$



## Partition of a set S

**Definition:** Let S be a set. A collection of nonempty subsets of S $A_1, A_2, \ldots, A_k$ is called **a partition of S** if:

- $A_i \cap A_j = \emptyset, \; i \neq j$ and $S = \bigcup_{i=1}^{k} A_i$

**Example:** Let S={1,2,3,4,5,6} and

- $A_1 = \{0,3,6\}$ $A_2 = \{1,4\}$ $A_3 = \{2,5\}$
- Is $A_1, A_{2,,} A_3$ a partition of S ?

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- Give a partition of S ?



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- $A_1 = \{0,3,6\}$ $A_2 = \{1,4\}$ $A_3 = \{2,5\}$
- Is $A_1, A_2, A_3$ a partition of S ? **Yes.**
- Give a partition of S ?
- {0,2,4,6} {1,3,5}
- {0} {1,2} {3,4,5} {6}

## Equivalence classes

**Theorem:** Let $R$ be an equivalence relation on a set $A$. The union of all the equivalence classes of $R$ is all of $A$, since an element $a$ of $A$ is in its own equivalence class $[a]_R$. In other words,

$$A = \bigcup_{a \in A} [a]_R$$



## Equivalence class

**Theorem:** Let R be an **equivalence relation** on a set A. The following statements are equivalent:

- i) a R b
- ii) [a] = [b]
- iii) $[a] \cap [b] \neq \varnothing$.

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**Proof:** (i) $\rightarrow$ (ii)

- Suppose **a R b**, i.e., $(a,b) \in R$. Want to show **[a] = [b].**
- Let $\mathbf{x} \in \mathbf{[a]} \rightarrow (a,x) \in R$.
- Since R is symmetric $(b,a) \in R$.
- Since R is transitive, $(b,a) \in R$ and $(a,x) \in R \rightarrow (b,x) \in R$. Thus, $x \in [b]$.
- Let $\mathbf{x} \in \mathbf{[b]} \rightarrow (b,x) \in R$.
- Since R is transitive, $(a,b) \in R$ and $(b,x) \in R \rightarrow (a,x) \in R$. Thus $x \in [a]$.
- Therefore [a] = [b].



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**Proof:** (ii) $\rightarrow$ (iii)

- Suppose [a] = [b]. Want to show $[a] \cap [b] \neq \varnothing$.
- Since R is reflexive, $a \in [a] \rightarrow [a] \neq \varnothing$ and the result follows.

## Equivalence class

**Theorem:** Let R be an **equivalence relation** on a set A. The following statements are equivalent:

- i) a R b
- ii) [a] = [b]
- iii) $[a] \cap [b] \neq \emptyset$.

**Proof:** (iii) $\rightarrow$ (i)

- Suppose $[a] \cap [b] \neq \emptyset$, want to show a R b.
- $[a] \cap [b] \neq \emptyset \rightarrow x \in [a] \cap [b] \rightarrow x \in [a]$ and $x \in [b] \rightarrow (a,x)$ and $(b,x) \in R$.
- Since R is symmetric $(x,b) \in R$. By the transitivity of R $(a,x) \in R$ and $(x,b) \in R$ implies $(a,b) \in R \rightarrow$ a R b.



## Partial orderings

**Definition:** A relation *R* on a set S is called a *partial ordering*, or *partial order*, if it is reflexive, antisymmetric, and transitive. A set together with a partial ordering *R* is called a *partially ordered set*, or *poset*, and is denoted by (*S*, *R*). Members of *S* are called *elements* of the poset.

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**Example:** Assume R denotes the "greater than or equal" relation ($\geq$) on the set S={1,2,3,4,5}.

- Is the relation reflexive?



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- Is the relation reflexive? Yes
- Is it antisymmetric?

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- Is the relation reflexive? Yes
- Is it antisymmetric? Yes
- Is it transitive?



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- Is it transitive? Yes
- **Conclusion:** R is a partial ordering.

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## Partial orderings

**Example:** Assume R is the divisibility relation (|) on the set of integers S={1,2,3,4,5,6}

- Is the relation reflexive? Yes
- Is it antisymmetric? Yes
- Is it transitive? Yes
- **Conclusion:** R is a partial ordering.