# CS 441 Discrete Mathematics for CS
# Lecture 22

# Relations III.

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# Composite of relations

**Definition:** Let R be a relation from a set A to a set B and S a relation from B to a set C. The **composite of R and S** is the relation consisting of the ordered pairs (a,c) where a $\in$ A and c $\in$ C, and for which there is a b $\in$ B such that (a,b) $\in$ R and (b,c) $\in$ S. We denote the composite of R and S by S o R.

**Examples:**

- Let A = {1,2,3}, B = {0,1,2} and C = {a,b}.
- R = {(1,0), (1,2), (3,1),(3,2)}
- S = {(0,b),(1,a),(2,b)}

- S o R = ?

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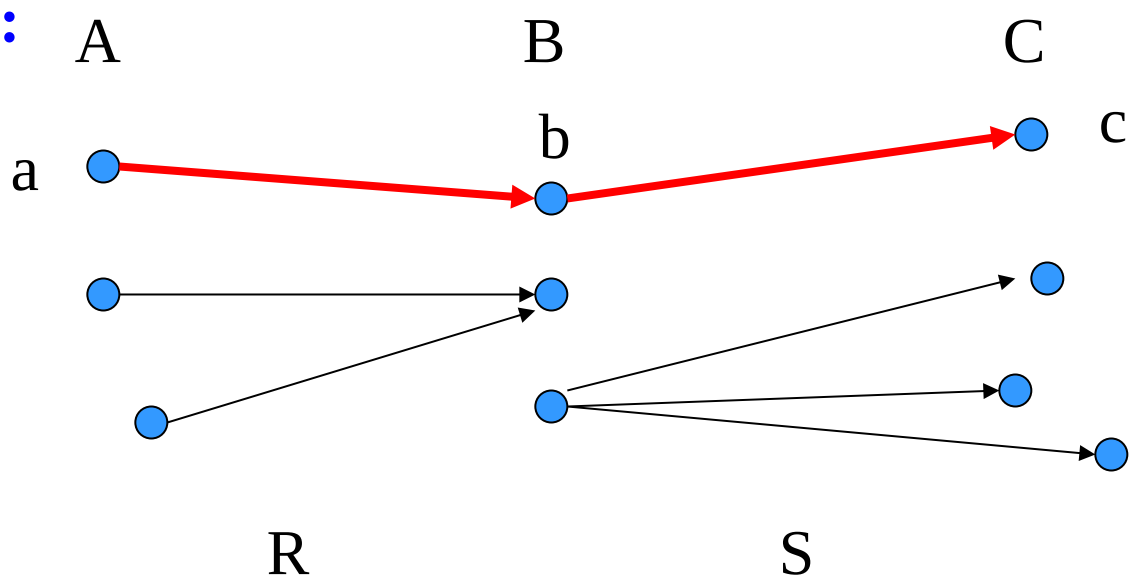


(a,c) ∈ S o R

# Composite of relations

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**Example:**

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- S o R = {(1,b),(3,a),(3,b)}

# Composite of relations

**Definition:** Let R be a relation on a set A. The **powers $R^n$**, n = 1,2,3,... is defined inductively by

- $\mathbf{R^1 = R}$ and $\mathbf{R^{n+1} = R^n \circ R}$.

**Examples**

- R = {(1,2),(2,3),(2,4), (3,3)} is a relation on A = {1,2,3,4}.
- $R^1 = R$ = {(1,2),(2,3),(2,4), (3,3)}
- $R^2$ = {(1,3), (1,4), (2,3), (3,3)}
- $R^3$ = {(1,3), (2,3), (3,3)}
- $R^4$ = {(1,3), (2,3), (3,3)}
- $R^k = R^3$, $k > 3$.

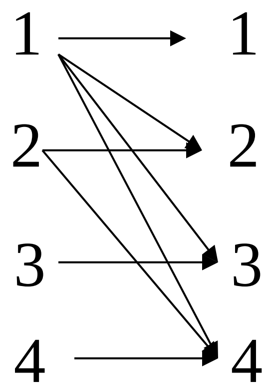
# Representing binary relations with graphs

- We can graphically represent a binary relation R from A to B as follows:
  - if **a R b** then draw an arrow from a to b.

$$\mathbf{a \rightarrow b}$$

**Example:**

- Relation $R_{div}$ (from previous lectures) on A={1,2,3,4}
- $R_{div}$ = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)}

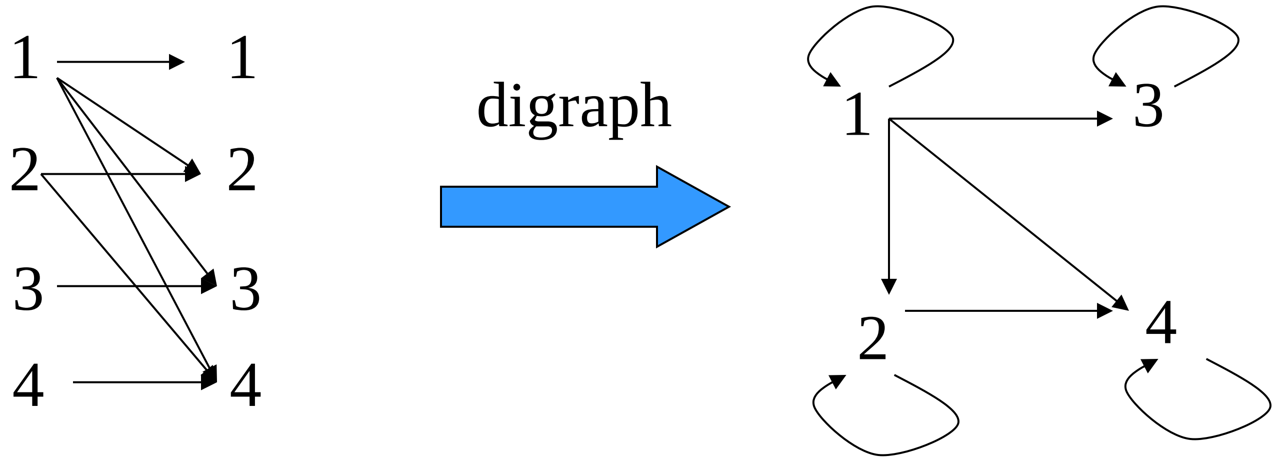


# Representing relations on a set with digraphs

**Definition:** A **directed graph or digraph** consists of a set of vertices (or nodes) together with a set E of ordered pairs of elements of V called edges (or arcs). The vertex a is called the initial vertex of the edge (a,b) and vertex b is the terminal vertex of this edge. An edge of the form (a,a) is called **a loop**.

**Example**

- Relation $R_{div}$ ={(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)}



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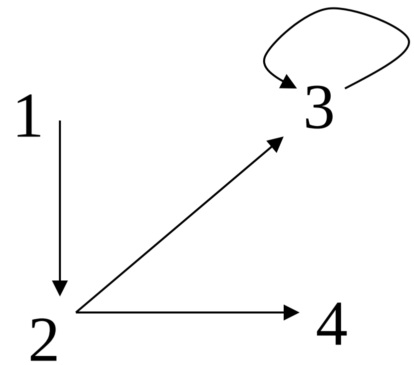


- $R^1 = R$ = {(1,2),(2,3),(2,4), (3,3)}
- $R^2$ = {(1,3), (1,4), (2,3), (3,3)}
- What does $R^2$ represent?
- Paths of length 2
- $R^3$ = {(1,3), (2,3), (3,3)}   path of length 3

# Transitive relation

**Definition (transitive relation):** A relation R on a set A is called **transitive** if

- $[(a,b) \in R \text{ and } (b,c) \in R] \rightarrow (a,c) \in R$ for all $a, b, c \in A$.

- **Example 1:**
- $R_{div} = \{(a\ b), \text{ if } a \mid b\}$ on $A = \{1,2,3,4\}$
- $R_{div} = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$
- **Is $R_{div}$ transitive?**
- **Answer: Yes.**

# Connection to $R^n$

**Theorem:** The relation R on a set A is transitive if and only if $R^n \subseteq R$ for n = 1,2,3,... .

**Proof: bi-conditional (if and only if)**

**( ← )** Suppose $R^n \subseteq R$, for n =1,2,3,... .

- Let $(a,b) \in R$ and $(b,c) \in R$
- by the definition of R o R, $(a,c) \in R \circ R \subseteq R \rightarrow$
- R is transitive.

# Connection to $R^n$

**Theorem:** The relation R on a set A is transitive **if and only if** $R^n \subseteq R$ for n = 1,2,3,... .

**Proof: biconditional (if and only if)**

(→) Suppose R is transitive. Show $R^n \subseteq R$, for n =1,2,3,... .

- Let P(n) : $R^n \subseteq R$. Math induction.
- **Basis Step:** P(1) says $R^1 = R$ so, $R^1 \subseteq R$ is true.
- **Inductive Step:** show $P(n) \rightarrow P(n+1)$
- Want to show if $R^n \subseteq R$ then $R^{n+1} \subseteq R$.
- Let $(a,b) \in R^{n+1}$ then by the definition of $R^{n+1} = R^n \circ R$ there is an element $x \in A$ so that $(a,x) \in R$ and $(x,b) \in R^n \subseteq R$ (inductive hypothesis). In addition to $(a,x) \in R$ and $(x,b) \in R$, R is transitive; so $(a,b) \in R$.
- Therefore, $R^{n+1} \subseteq R$.

# Closures of relations

- Let R={(1,1),(1,2),(2,1),(3,2)} on A ={1 2 3}.
- Is this relation reflexive?
- Answer: **No.** Why?
- **(2,2) and (3,3) is not in R.**

- The question is what is **the minimal relation S $\supseteq$** R that is reflexive?
- How to make R reflexive with minimum number of additions?
- **Answer:** Add (2,2) and (3,3)
  - Then S= {(1,1),(1,2),(2,1),(3,2),(2,2),(3,3)}
  - R $\subseteq$ S
  - The minimal set S $\supseteq$ R is called **the reflexive closure of R**

# Reflexive closure

The set S is called **the reflexive closure of R** if it:

- contains R
- has reflexive property
- is contained in every reflexive relation Q that contains R ($R \subseteq Q$), that is $S \subseteq Q$

# Closures on relations

- **Relations can have different properties:**
  - **reflexive,**
  - **symmetric**
  - **transitive**

- Because of that we can have:
  - **symmetric,**
  - **reflexive and**
  - **transitive**

  **closures.**

# Closures

**Definition:** Let R be a relation on a set A. A relation S on A with property P is called **the closure of R with respect to P** if S is a subset of every relation Q ($S \subseteq Q$) with property P that contains R ($R \subseteq Q$).

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**Example (a symmetric closure):**

- Assume R={(1,2),(1,3), (2,2)} on A={1,2,3}.
- What is the symmetric closure S of R?
- S = {(1,2),(1,3), (2,2)} $\cup$ **{(**2,1), (3,1)}

  = {(1,2),(1,3), (2,2),(2,1), (3,1)}

# Closures

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**Example (transitive closure):**

- Assume R={(1,2), (2,2), (2,3)} on A={1,2,3}.
- **Is R transitive? No.**
- **How to make it transitive?**
- **S = {(1,2), (2,2), (2,3)} $\cup$ {(1,3)}**

  **= {(1,2), (2,2), (2,3),(1,3)}**
- S is the transitive closure of R

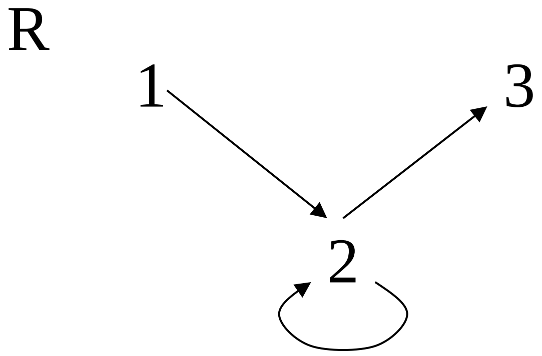
# Transitive closure

We can represent the relation on the graph. Finding a transitive closure corresponds to finding all pairs of elements that are connected with a directed path (or digraph).

**Example:**

Assume R={(1,2), (2,2), (2,3)} on A={1,2,3}.

Transitive closure S = {(1,2), (2,2), (2,3),(1,3)}.



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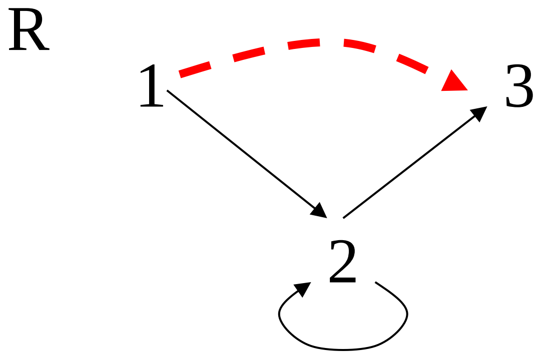

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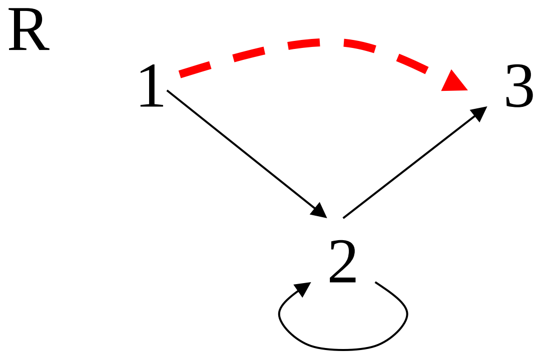
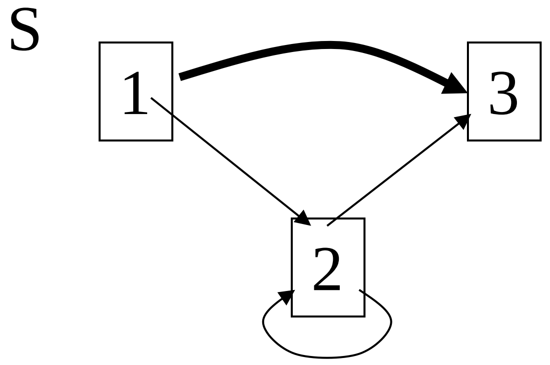
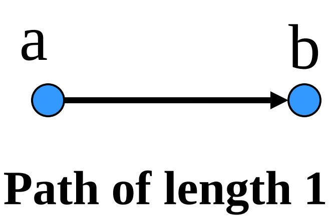


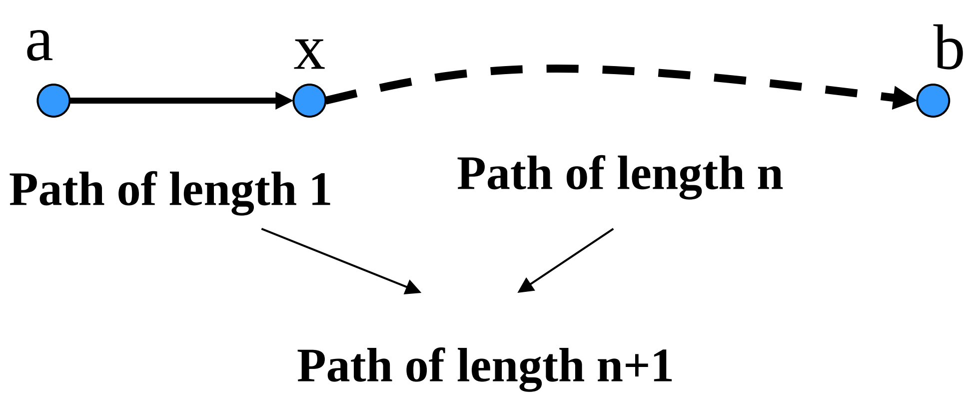



# Transitive closure

**Theorem:** Let R be a relation on a set A. There is a path of length n from a to b if and only if $(a,b) \in R^n$.

**Proof (math induction):**



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**Proof (math induction):**

- **P(1):** There is a path of length 1 from a to b if and only if $(a,b) \in R^1$, by the definition of R.
- **Show $P(n) \rightarrow P(n+1)$:** Assume there is a path of length n from a to b if and only if $(a,b) \in R^n \rightarrow$ there is a path of length n+1 from a to b if and only if $(a,b) \in R^{n+1}$.
- There is a path of length n+1 from a to b if and only if there exists an $x \in A$, such that $(a,x) \in R$ (a path of length 1) and $(x,b) \in R^n$ is a path of length n from x to b.

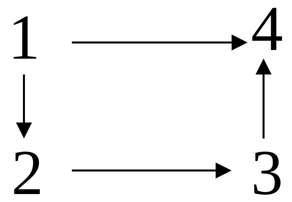

- $(x,b) \in R^n$ holds due to P(n). Therefore, there is a path of length n + 1 from a to b. This also implies that $(a,b) \in R^{n+1}$.

# Connectivity relation

**Definition:** Let R be a relation on a set A. The **connectivity relation** R* consists of all pairs (a,b) such that there is a path (of any length, ie. 1 or 2 or 3 or ...) between a and b in R.

$$R^* = \bigcup_{k=1}^{\infty} R^k$$

**Example:**




- A = {1,2,3,4}
- R = {(1,2),(1,4),(2,3),(3,4)}
- $R^2$ = {(1,3),(2,4)}
- $R^3$ = {(1,4)}
- $R^4 = \emptyset$
- ...
- R* = {(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)}

# Transitivity closure and connectivity relation

**Theorem:** The transitive closure of a relation R **equals** the connectivity relation R*.

Based on the following **Lemma.**

**Lemma 1:** Let A be a set with n elements, and R a relation on A. If there is a path from a to b, then there exists a path of length < n in between (a,b). Consequently:

$$R^* = \bigcup_{k=1}^{n} R^k$$

# Connectivity

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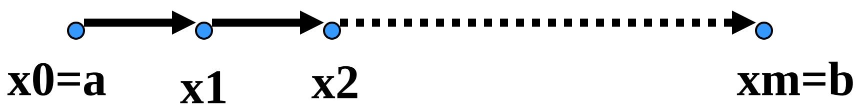
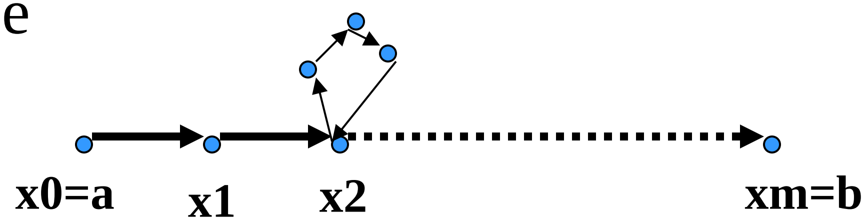
**Proof (intuition):**

- There are at most n different elements we can visit on a path if the path does not have loops




- Loops may increase the length but the same node is visited more than once



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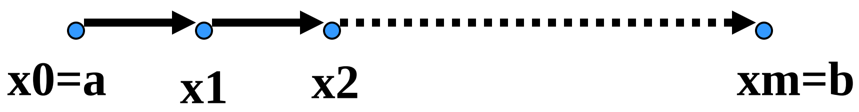


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