CS 441 Discrete Mathematics for CS
Lecture 21

Relations II

Milos Hauskrecht

milos@cs.pitt.edu

5329 Sennott Square

Course administration

- **Homework assignment 9**
  - **due on Monday, April 8, 2013**

Course web page:

http://www.cs.pitt.edu/~milos/courses/cs441/

Cartesian product (review)

- Let A={a_1, a_2, ..a_k} and B={b_1,b_2,..b_m}.
- **The Cartesian product** A x B is defined by a set of pairs {(a_1 b_1), (a_1, b_2), … (a_1, b_m), …, (a_k,b_m)}.

Example:

Let A={a,b,c} and B={1 2 3}. What is AxB?

AxB = {(a,1),(a,2),(a,3),(b,1),(b,2),(b,3)}

Binary relation

<u>Definition:</u> Let A and B be sets. A **binary relation from A to B** is **a subset of a Cartesian product A x B**.

Example: Let A={a,b,c} and B={1,2,3}.

- R={(a,1),(b,2),(c,2)} is an example of a relation from A to B.

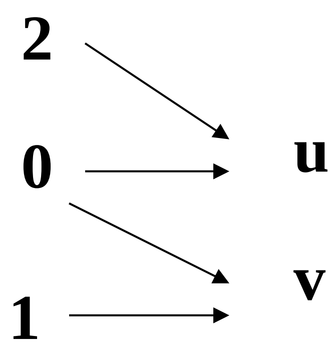
Representing binary relations

- We can graphically represent a binary relation R as follows:
  - if **a R b** then draw an arrow from a to b.

$$\mathbf{a \rightarrow b}$$

Example:

- Let A = {0, 1, 2}, B = {u,v} and R = { (0,u), (0,v), (1,v), (2,u) }
- Note: $R \subseteq A \times B$.
- **Graph:**



Representing binary relations

- We can represent a binary relation R by a **table** showing (marking) the ordered pairs of R.

Example:

- Let A = {0, 1, 2}, B = {u,v} and R = { (0,u), (0,v), (1,v), (2,u) }
- **Table:**

R	u	v
0	x	x
1		x
2	x	

or

R	u	v
0	1	1
1	0	1
2	1	0

Properties of relations

Properties of relations on A:

- Reflexive
- Irreflexive
- Symmetric
- Anti-symmetric

Reflexive relation

Reflexive relation

- R_{div} ={(a b), if a |b} on A = {1,2,3,4}
- R_{div} = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)}

$$MR_{div} = \begin{matrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}$$

- **A relation R is reflexive** if and only if MR has 1 in every position on its main diagonal.

Irreflexive relation

Irreflexive relation

- $R_{\neq}$ on A={1,2,3,4}, such that **a $R_{\neq}$ b** if and only if $a \neq b$.
- **$R_{\neq}$={(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,4),(4,1),(4,2),(4,3)}**

$$MR = \begin{matrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{matrix}$$

- **A relation R is irreflexive** if and only if MR has 0 in every position on its main diagonal.

Symmetric relation

Symmetric relation:

- $R_{\neq}$ on A={1,2,3,4}, such that **a $R_{\neq}$ b** if and only if a ≠ b.
- **$R_{\neq}$={(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,4),(4,1),(4,2),(4,3)}**

$$MR = \begin{matrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{matrix}$$

- **A relation R is symmetric** if and only if $m_{ij} = m_{ji}$ for all i,j.

Antisymmetric relations

Antisymmetric relation

- relation $R_{fun} = \{(1,2),(2,2),(3,3)\}$

$$MR_{fun} = \begin{matrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}$$

- A relation is **antisymmetric** if and only if $m_{ij} = 1 \rightarrow m_{ji} = 0$ for $i \neq j$.

Properties of relations

Definition (transitive relation): A relation R on a set A is called **transitive** if

- $[(a,b) \in R \text{ and } (b,c) \in R] \rightarrow (a,c) \in R$ for all $a, b, c \in A$.

- **Example 2**:
- $R_{\neq}$ on A={1,2,3,4}, such that **a $R_{\neq}$ b** if and only if $a \neq b$.
- **$R_{\neq}$={(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,4),(4,1),(4,2),(4,3)}**
- **Is $R_{\neq}$ transitive?**
- **Answer: No.** It is not transitive since $(1,2) \in R$ and $(2,1) \in R$ but (1,1) is not an element of R.

Properties of relations

Definition (transitive relation): A relation R on a set A is called **transitive** if

- $[(a,b) \in R \text{ and } (b,c) \in R] \rightarrow (a,c) \in R$ for all $a, b, c \in A$.

- **Example** 3:
- Relation R_{fun} on A = {1,2,3,4} defined as:
  - $R_{fun} = \{(1,2),(2,2),(3,3)\}$.
- **Is R_{fun} transitive?**
- **Answer: Yes.** It is transitive.

Combining relations

Definition: Let A and B be sets. A **binary relation from A to B** is a subset of a Cartesian product A x B.

- Let $R \subseteq A \times B$ means R is a set of ordered pairs of the form (a,b) where $a \in A$ and $b \in B$.

Combining Relations

- **Relations are sets → combinations via set operations**
- Set operations of: **union, intersection, difference and symmetric difference.**

Combining relations

Example:

- Let A = {1,2,3} and B = {u,v} and
- R1 = {(1,u), (2,u), (2,v), (3,u)}
- R2 = {(1,v),(3,u),(3,v)}

What is:

- $R1 \cup R2$ = {(1,u),(1,v),(2,u),(2,v),(3,u),(3,v)}
- $R1 \cap R2$ = {(3,u)}
- R1 - R2 = {(1,u),(2,u),(2,v)}
- R2 - R1 = {(1,v),(3,v)}

Combination of relations: implementation

Definition. The **join,** denoted by ∨, of two m-by-n matrices (a_{ij}) and (b_{ij}) of 0s and 1s is an m-by-n matrix (m_{ij}) where

- $m_{ij} = a_{ij} \vee b_{ij}$ for all i,j

= pairwise or (disjunction)

- **Example**:
- Let A = {1,2,3} and B = {u,v} and
- R1 = {(1,u), (2,u), (2,v), (3,u)}
- R2 = {(1,v),(3,u),(3,v)}

- **MR1 =** $\begin{matrix} 1 & 0 \\ 1 & 1 \\ 1 & 0 \end{matrix}$ **MR2 =** $\begin{matrix} 0 & 1 \\ 0 & 0 \\ 1 & 1 \end{matrix}$ **M(R1 ∨ R2)=** $\begin{matrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{matrix}$

Combination of relations: implementation

Definition. The **meet,** denoted by $\wedge$, of two m-by-n matrices (a_{ij}) and (b_{ij}) of 0s and 1s is an m-by-n matrix (m_{ij}) where

- $m_{ij} = a_{ij} \wedge b_{ij}$ for all i,j

  = **pairwise and (conjunction)**

- **Example**:
- Let A = {1,2,3} and B = {u,v} and
- R1 = {(1,u), (2,u), (2,v), (3,u)}
- R2 = {(1,v),(3,u),(3,v)}

- **MR1 =** $\begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{bmatrix}$ **MR2 =** $\begin{bmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} \end{bmatrix}$ **MR1** $\wedge$ **MR2=** $\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} \end{bmatrix}$

Composite of relations

Definition: Let R be a relation from a set A to a set B and S a relation from B to a set C. The **composite of R and S** is the relation consisting of the ordered pairs (a,c) where $a \in A$ and $c \in C$, and for which there is a $b \in B$ such that $(a,b) \in R$ and $(b,c) \in S$. We denote the composite of R and S by S o R.

Examples:

- Let A = {1,2,3}, B = {0,1,2} and C = {a,b}.
- R = {(1,0), (1,2), (3,1),(3,2)}
- S = {(0,b),(1,a),(2,b)}

- S o R = {(1,b),(3,a),(3,b)}

Implementation of composite

Definition. The **Boolean product**, **denoted by ⊙,** of an m-by-n matrix (a_{ij}) and n-by-p matrix (b_{jk}) of 0s and 1s is an m-by-p matrix (m_{ik}) where

- m_{ik} = 1, if a_{ij} = 1 and b_{jk} = 1 for some k=1,2,...,n
  0, otherwise

Examples:

- Let A = {1,2,3}, B = {0,1,2} and C = {a,b}.
- R = {(1,0), (1,2), (3,1),(3,2)}
- S = {(0,b),(1,a),(2,b)}

- S o R = {(1,b),(3,a),(3,b)}

Implementation of composite

Examples:

- Let A = {1,2}, {1,2,3} C = {a,b}
- R = {(1,2),(1,3),(2,1)} is a relation from A to B
- S = {(1,a),(3,b),(3,a)} is a relation from B to C.
- S ○ R = {(1,b),(1,a),(2,a)}

$$M_R = \begin{matrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{matrix} \qquad M_S = \begin{matrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{matrix}$$

$$M_R \odot M_S = \begin{matrix} x & x \\ x & x \end{matrix}$$

Implementation of composite

Examples:

- Let A = {1,2}, {1,2,3} C = {a,b}
- R = {(1,2),(1,3),(2,1)} is a relation from A to B
- S = {(1,a),(3,b),(3,a)} is a relation from B to C.
- S ○ R = {(1,b),(1,a),(2,a)}

$$M_R = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad M_S = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

$$M_R \odot M_S = \begin{bmatrix} \mathbf{1} & x \\ x & x \end{bmatrix}$$

Implementation of composite

Examples:

- Let A = {1,2}, {1,2,3} C = {a,b}
- R = {(1,2),(1,3),(2,1)} is a relation from A to B
- S = {(1,a),(3,b),(3,a)} is a relation from B to C.
- S ○ R = {(1,b),(1,a),(2,a)}

$$M_R = \begin{array}{ccc} 0 & 1 & 1 \\ 1 & 0 & 0 \end{array} \qquad M_S = \begin{array}{cc} 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{array}$$

$$M_R \odot M_S = \begin{array}{cc} 1 & \mathbf{1} \\ x & x \end{array}$$

Implementation of composite

Examples:

- Let A = {1,2}, {1,2,3} C = {a,b}
- R = {(1,2),(1,3),(2,1)} is a relation from A to B
- S = {(1,a),(3,b),(3,a)} is a relation from B to C.
- $S \circ R$ = {(1,b),(1,a),(2,a)}

$$M_R = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad M_S = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

$$M_R \odot M_S = \begin{bmatrix} 1 & 1 \\ 1 & x \end{bmatrix}$$

Implementation of composite

Examples:

- Let A = {1,2}, {1,2,3} C = {a,b}
- R = {(1,2),(1,3),(2,1)} is a relation from A to B
- S = {(1,a),(3,b),(3,a)} is a relation from B to C.
- S ○ R = {(1,b),(1,a),(2,a)}

$$M_R = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad M_S = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

$$M_R \odot M_S = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$M_{S \circ R} = \ ?$$

Implementation of composite

Examples:

- Let A = {1,2}, {1,2,3} C = {a,b}
- R = {(1,2),(1,3),(2,1)} is a relation from A to B
- S = {(1,a),(3,b),(3,a)} is a relation from B to C.
- S ○ R = {(1,b),(1,a),(2,a)}

$$M_R = \begin{matrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{matrix} \qquad M_S = \begin{matrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{matrix}$$

$$M_R \odot M_S = \begin{matrix} 1 & 1 \\ 1 & 0 \end{matrix}$$

$$M_{S \circ R} = \begin{matrix} 1 & 1 \\ 1 & 0 \end{matrix}$$

Composite of relations

Definition: Let R be a relation on a set A. The **powers R^n**, n = 1,2,3,... is defined inductively by

- $\mathbf{R^1 = R}$ and $\mathbf{R^{n+1} = R^n \circ R}$.

Examples

- R = {(1,2),(2,3),(2,4), (3,3)} is a relation on A = {1,2,3,4}.
- $R^1 = R$ = {(1,2),(2,3),(2,4), (3,3)}
- R^2 = {(1,3), (1,4), (2,3), (3,3)}
- R^3 = {(1,3), (2,3), (3,3)}
- R^4 = {(1,3), (2,3), (3,3)}
- $R^k = R^3$, $k > 3$.