Relations II

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Course administration

- Homework assignment 9
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Course web page:

http://www.cs.pitt.edu/~milos/courses/cs441/
Cartesian product (review)

- Let $A=\{a_1, a_2, \ldots a_k\}$ and $B=\{b_1, b_2, \ldots b_m\}$.
- **The Cartesian product** $A \times B$ is defined by a set of pairs 
  \{(a_1 b_1), (a_1, b_2), \ldots (a_1, b_m), \ldots, (a_k, b_m)\}.

**Example:**
Let $A=\{a, b, c\}$ and $B=\{1, 2, 3\}$. What is $A \times B$?

$$A \times B = \{(a,1),(a,2),(a,3),(b,1),(b,2),(b,3)\}$$
Binary relation

**Definition:** Let $A$ and $B$ be sets. A **binary relation from $A$ to $B$** is a subset of a Cartesian product $A \times B$.

**Example:** Let $A=\{a, b, c\}$ and $B=\{1, 2, 3\}$.

- $R=\{(a,1),(b,2),(c,2)\}$ is an example of a relation from $A$ to $B$. 
Representing binary relations

- We can graphically represent a binary relation $R$ as follows:
  - if $a \, R \, b$ then draw an arrow from $a$ to $b$.

$$a \rightarrow b$$

**Example:**
- Let $A = \{0, 1, 2\}$, $B = \{u, v\}$ and $R = \{(0,u), (0,v), (1,v), (2,u)\}$
- Note: $R \subseteq A \times B$.
- **Graph:**

```
  2
 /|
/  |
0  u
 /|
 /  |
/   |
1  v
```
Representing binary relations

- We can represent a binary relation $R$ by a **table** showing (marking) the ordered pairs of $R$.

**Example:**
- Let $A = \{0, 1, 2\}$, $B = \{u, v\}$ and  $R = \{ (0,u), (0,v), (1,v), (2,u) \}$
- Table:

<table>
<thead>
<tr>
<th>R</th>
<th>u</th>
<th>v</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>x</td>
<td>x</td>
</tr>
<tr>
<td>1</td>
<td>x</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>x</td>
<td></td>
</tr>
</tbody>
</table>

  or

<table>
<thead>
<tr>
<th>R</th>
<th>u</th>
<th>v</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>
Properties of relations

**Properties of relations on A:**

- Reflexive
- Irreflexive
- Symmetric
- Anti-symmetric
Reflexive relation

• $R_{\text{div}} = \{(a, b), \text{ if } a | b\}$ on $A = \{1,2,3,4\}$
• $R_{\text{div}} = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$

\[
\begin{array}{cccccc}
1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}
\]

• A relation $R$ is reflexive if and only if $MR$ has 1 in every position on its main diagonal.
Irreflexive relation

Irreflexive relation

• $R_\neq$ on $A=\{1,2,3,4\}$, such that $a \, R_\neq \, b$ if and only if $a \neq b$.

• $R_\neq=\{(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,4),(4,1),(4,2),(4,3)\}$

\[
\begin{array}{cccc}
0 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 \\
1 & 1 & 1 & 0 \\
\end{array}
\]

\[MR = \begin{array}{cccc}
1 & 1 & 0 & 1 \\
1 & 1 & 1 & 0 \\
\end{array}\]

• **A relation $R$ is irreflexive** if and only if $MR$ has 0 in every position on its main diagonal.
Symmetric relation

Symmetric relation:

- \( R \neq \) on \( A=\{1,2,3,4\} \), such that \( a R \neq b \) if and only if \( a \neq b \).
- \( R=\{(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,4),(4,1),(4,2),(4,3)\} \)

\[
\begin{array}{cccc}
0 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 \\
1 & 1 & 1 & 0 \\
\end{array}
\]

\( MR = \)

- A relation \( R \) is symmetric if and only if \( m_{ij} = m_{ji} \) for all \( i,j \).
Antisymmetric relations

**Antisymmetric relation**

- relation \( R_{\text{fun}} = \{(1,2),(2,2),(3,3)\} \)

\[
\begin{array}{cccc}
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
\end{array}
\]

- A relation is **antisymmetric** if and only if \( m_{ij} = 1 \rightarrow m_{ji} = 0 \) for \( i \neq j \).
Properties of relations

**Definition (transitive relation):** A relation $R$ on a set $A$ is called transitive if

- $[(a,b) \in R \text{ and } (b,c) \in R] \implies (a,c) \in R$ for all $a, b, c \in A$.

**Example 2:**

- $R \neq$ on $A=\{1,2,3,4\}$, such that $a \mathcal{R} \neq b$ if and only if $a \neq b$.
- $R \neq =\{(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,4),(4,1),(4,2),(4,3)\}$
- **Is $R \neq$ transitive?**
- **Answer:** No. It is not transitive since $(1,2) \in R$ and $(2,1) \in R$ but $(1,1)$ is not an element of $R$. 
Properties of relations

Definition (transitive relation): A relation $R$ on a set $A$ is called transitive if

- $[(a,b) \in R \text{ and } (b,c) \in R] \rightarrow (a,c) \in R \text{ for all } a, b, c \in A.$

Example 3:
- Relation $R_{\text{fun}}$ on $A = \{1,2,3,4\}$ defined as:
  - $R_{\text{fun}} = \{(1,2),(2,2),(3,3)\}.$
- Is $R_{\text{fun}}$ transitive?
- Answer: Yes. It is transitive.
Combining relations

**Definition:** Let A and B be sets. A *binary relation from A to B* is a subset of a Cartesian product $A \times B$.

- Let $R \subseteq A \times B$ means $R$ is a set of ordered pairs of the form $(a,b)$ where $a \in A$ and $b \in B$.

**Combining Relations**

- Relations are sets $\rightarrow$ combinations via set operations
- Set operations of: union, intersection, difference and symmetric difference.
Combining relations

Example:

• Let $A = \{1,2,3\}$ and $B = \{u,v\}$ and
• $R_1 = \{(1,u), (2,u), (2,v), (3,u)\}$
• $R_2 = \{(1,v),(3,u),(3,v)\}$

What is:

• $R_1 \cup R_2 = \{(1,u),(1,v),(2,u),(2,v),(3,u),(3,v)\}$
• $R_1 \cap R_2 = \{(3,u)\}$
• $R_1 - R_2 = \{(1,u),(2,u),(2,v)\}$
• $R_2 - R_1 = \{(1,v),(3,v)\}$
Combination of relations: implementation

**Definition.** The *join*, denoted by $\lor$, of two $m$-by-$n$ matrices $(a_{ij})$ and $(b_{ij})$ of 0s and 1s is an $m$-by-$n$ matrix $(m_{ij})$ where

- $m_{ij} = a_{ij} \lor b_{ij}$ for all $i,j$

= pairwise or (disjunction)

**Example:**
- Let $A = \{1,2,3\}$ and $B = \{u,v\}$ and
- $R1 = \{(1,u), (2,u), (2,v), (3,u)\}$
- $R2 = \{(1,v),(3,u),(3,v)\}$

<table>
<thead>
<tr>
<th>$M(R1 \lor R2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 1 1 1</td>
</tr>
<tr>
<td>1 1 0 0</td>
</tr>
<tr>
<td>1 0 1 1</td>
</tr>
</tbody>
</table>

$MR1 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

$MR2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$
Combination of relations: implementation

**Definition.** The **meet**, denoted by $\land$, of two m-by-n matrices $(a_{ij})$ and $(b_{ij})$ of 0s and 1s is an m-by-n matrix $(m_{ij})$ where

- $m_{ij} = a_{ij} \land b_{ij}$ for all $i,j$
- = pairwise and (conjunction)

**Example:**

- Let $A = \{1,2,3\}$ and $B = \{u,v\}$ and
- $R_1 = \{(1,u), (2,u), (2,v), (3,u)\}$
- $R_2 = \{(1,v),(3,u),(3,v)\}$

<table>
<thead>
<tr>
<th></th>
<th>$R_1$</th>
<th>$R_2$</th>
<th>$R_1 \land R_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
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</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

1 0 1 1
**Composite of relations**

**Definition:** Let $R$ be a relation from a set $A$ to a set $B$ and $S$ a relation from $B$ to a set $C$. The **composite of $R$ and $S$** is the relation consisting of the ordered pairs $(a,c)$ where $a \in A$ and $c \in C$, and for which there is a $b \in B$ such that $(a,b) \in R$ and $(b,c) \in S$. We denote the composite of $R$ and $S$ by $S \circ R$.

**Examples:**

- Let $A = \{1,2,3\}$, $B = \{0,1,2\}$ and $C = \{a,b\}$.
- $R = \{(1,0), (1,2), (3,1),(3,2)\}$
- $S = \{(0,b),(1,a),(2,b)\}$
- $S \circ R = \{(1,b),(3,a),(3,b)\}$
Implementation of composite

**Definition.** The **Boolean product**, denoted by $\odot$, of an $m$-by-$n$ matrix $(a_{ij})$ and $n$-by-$p$ matrix $(b_{jk})$ of 0s and 1s is an $m$-by-$p$ matrix $(m_{ik})$ where

- $m_{ik} = 1$, if $a_{ij} = 1$ and $b_{jk} = 1$ for some $k=1,2,...,n$
- $0$, otherwise

**Examples:**

- Let $A = \{1,2,3\}$, $B = \{0,1,2\}$ and $C = \{a,b\}$.
- $R = \{(1,0), (1,2), (3,1),(3,2)\}$
- $S = \{(0,b),(1,a),(2,b)\}$

- $S \circ R = \{(1,b),(3,a),(3,b)\}$
Implementation of composite

Examples:

- Let $A = \{1, 2\}, \{1, 2, 3\}$ $C = \{a, b\}$
- $R = \{(1, 2), (1, 3), (2, 1)\}$ is a relation from $A$ to $B$
- $S = \{(1, a), (3, b), (3, a)\}$ is a relation from $B$ to $C$.
- $S \circ R = \{(1, b), (1, a), (2, a)\}$

\[
\begin{array}{ccc}
0 & 1 & 1 \\
M_R = 1 & 0 & 0 \\
\end{array}
\quad
\begin{array}{ccc}
1 & 0 \\
M_S = 0 & 0 \\
\end{array}
\quad
\begin{array}{ccc}
x & x \\
M_R \odot M_S = x & x \\
\end{array}
\quad
\begin{array}{ccc}
x & x \\
\end{array}
\]
Implementation of composite

Examples:

• Let $A = \{1,2\}, \{1,2,3\}$ $C = \{a,b\}$
• $R = \{(1,2),(1,3),(2,1)\}$ is a relation from $A$ to $B$
• $S = \{(1,a),(3,b),(3,a)\}$ is a relation from $B$ to $C$.
• $S \circ R = \{(1,b),(1,a),(2,a)\}$

\[
\begin{bmatrix}
0 & 1 & 1 \\
\end{bmatrix}
\]

\[
M_R = \begin{bmatrix}
1 & 0 & 0 \\
\end{bmatrix}
\]

\[
M_S = \begin{bmatrix}
0 & 0 \\
1 & 1 \\
\end{bmatrix}
\]

\[
M_R \odot M_S = \begin{bmatrix}
1 & x \\
x & x \\
\end{bmatrix}
\]
Implementation of composite

Examples:

• Let $A = \{1,2\}, \{1,2,3\}$ $C = \{a,b\}$
• $R = \{(1,2),(1,3),(2,1)\}$ is a relation from $A$ to $B$
• $S = \{(1,a),(3,b),(3,a)\}$ is a relation from $B$ to $C.$
• $S \circ R = \{(1,b),(1,a),(2,a)\}$

\[
M_R = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad M_S = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad M_R \odot M_S = \begin{bmatrix} 1 & 1 \\ x & x \end{bmatrix}
\]
Implementation of composite

Examples:

• Let \( A = \{1,2\}, \{1,2,3\} \) \( C = \{a,b\} \)
• \( R = \{(1,2),(1,3),(2,1)\} \) is a relation from \( A \) to \( B \)
• \( S = \{(1,a),(3,b),(3,a)\} \) is a relation from \( B \) to \( C \).
• \( S \circ R = \{(1,b),(1,a),(2,a)\} \)

\[
M_R = \begin{bmatrix}
0 & 1 & 1 \\
1 & 0 & 0
\end{bmatrix} \quad M_S = \begin{bmatrix}
1 & 0 \\
0 & 0 \\
1 & 1
\end{bmatrix}
\]

\[
M_R \circ M_S = \begin{bmatrix}
1 & 1 \\
1 & x
\end{bmatrix}
\]
Implementation of composite

Examples:

• Let $A = \{1,2\}, \{1,2,3\}$ $C = \{a,b\}$
• $R = \{(1,2),(1,3),(2,1)\}$ is a relation from $A$ to $B$
• $S = \{(1,a),(3,b),(3,a)\}$ is a relation from $B$ to $C$.
• $S \circ R = \{(1,b),(1,a),(2,a)\}$

\[
M_R = \begin{bmatrix}
0 & 1 & 1 \\
1 & 0 & 0 \\
\end{bmatrix}, \quad M_S = \begin{bmatrix}
1 & 0 \\
0 & 0 \\
1 & 1 \\
\end{bmatrix}
\]

\[
M_R \odot M_S = \begin{bmatrix}
1 & 1 \\
1 & 0 \\
\end{bmatrix}, \quad M_S \odot R = ?
\]
Implementation of composite

Examples:
• Let \( A = \{1,2\}, \{1,2,3\} \) \( C = \{a,b\} \)
• \( R = \{(1,2),(1,3),(2,1)\} \) is a relation from \( A \) to \( B \)
• \( S = \{(1,a),(3,b),(3,a)\} \) is a relation from \( B \) to \( C \).
• \( S \circ R = \{(1,b),(1,a),(2,a)\} \)

\[
\begin{bmatrix}
0 & 1 & 1 & 1 & 0
\end{bmatrix}
\]

\[
M_R = \begin{bmatrix}
1 & 0 & 0 & M_S & = & 0 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 1
\end{bmatrix}
\]

\[
M_R \odot M_S = \begin{bmatrix}
1 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0
\end{bmatrix}
\]

\[
M_S \circ R = \begin{bmatrix}
1 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0
\end{bmatrix}
\]

Composite of relations

**Definition:** Let $R$ be a relation on a set $A$. The powers $R^n$, $n = 1, 2, 3, \ldots$ is defined inductively by

- $R^1 = R$ and $R^{n+1} = R^n \circ R$.

**Examples**

- $R = \{(1,2),(2,3),(2,4), (3,3)\}$ is a relation on $A = \{1,2,3,4\}$.
- $R^1 = R = \{(1,2),(2,3),(2,4), (3,3)\}$
- $R^2 = \{(1,3), (1,4), (2,3), (3,3)\}$
- $R^3 = \{(1,3), (2,3), (3,3)\}$
- $R^4 = \{(1,3), (2,3), (3,3)\}$
- $R^k = R^3$, $k > 3$. 