Relations

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Cartesian product (review)

Example:
Let A={a,b,c} and B={1 2 3}. What is AxB?
AxB = {(a,1),(a,2),(a,3),(b,1),(b,2),(b,3)}
**Cartesian product (review)**

Let $A=\{a_1, a_2, \ldots, a_k\}$ and $B=\{b_1, b_2, \ldots, b_m\}$.

**The Cartesian product** $A \times B$ is defined by a set of pairs $\{(a_1, b_1), (a_1, b_2), \ldots, (a_1, b_m), \ldots, (a_k, b_m)\}$.

**Example:**

Let $A=\{a, b, c\}$ and $B=\{1, 2, 3\}$. What is $A \times B$?

$A \times B = \{(a,1), (a,2), (a,3), (b,1), (b,2), (b,3)\}$

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**Binary relation**

**Definition:** Let $A$ and $B$ be two sets. A **binary relation from $A$ to $B$** is a subset of a Cartesian product $A \times B$.

- Let $R \subseteq A \times B$ means $R$ is a set of ordered pairs of the form $(a,b)$ where $a \in A$ and $b \in B$.
- We use the notation $a R b$ to denote $(a,b) \in R$ and $a \not R b$ to denote $(a,b) \not \in R$. If $a R b$, we say $a$ is related to $b$ by $R$.

**Example:** Let $A=\{a,b,c\}$ and $B=\{1,2,3\}$.

- Is $R=\{(a,1),(b,2),(c,2)\}$ a relation from $A$ to $B$? **Yes**.
- Is $Q=\{(1,a),(2,b)\}$ a relation from $A$ to $B$? **No**.
- Is $P=\{(a,a),(b,c),(b,a)\}$ a relation from $A$ to $A$?
**Binary relation**

**Definition:** Let A and B be two sets. A **binary relation from A to B** is a subset of a Cartesian product A x B.

- Let $R \subseteq A \times B$ means $R$ is a set of ordered pairs of the form $(a, b)$ where $a \in A$ and $b \in B$.
- We use the notation $a \mathrel{R} b$ to denote $(a, b) \in R$ and $a \not\mathrel{R} b$ to denote $(a, b) \notin R$. If $a \mathrel{R} b$, we say a is related to b by $R$.

**Example:** Let $A = \{a, b, c\}$ and $B = \{1, 2, 3\}$.
- Is $R = \{(a, 1), (b, 2), (c, 2)\}$ a relation from A to B? **Yes**.
- Is $Q = \{(1, a), (2, b)\}$ a relation from A to B? **No**.
- Is $P = \{(a, a), (b, c), (b, a)\}$ a relation from A to A? **Yes**.

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**Representing binary relations**

- We can graphically represent a binary relation $R$ as follows:
  - if $a \mathrel{R} b$ then draw an arrow from a to b.
    $$a \rightarrow b$$

**Example:**
- Let $A = \{0, 1, 2\}$, $B = \{u, v\}$ and $R = \{(0, u), (0, v), (1, v), (2, u)\}$
- Note: $R \subseteq A \times B$.
- **Graph:**

```
    2
   /|\
  0-|--|-
   v  u
   1
```
Representing binary relations

- We can represent a binary relation $R$ by a table showing (marking) the ordered pairs of $R$.

Example:
- Let $A = \{0, 1, 2\}$, $B = \{u, v\}$ and $R = \{(0, u), (0, v), (1, v), (2, u)\}$
- Table:

<table>
<thead>
<tr>
<th></th>
<th>u</th>
<th>v</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>x</td>
<td>x</td>
</tr>
<tr>
<td>1</td>
<td>x</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>x</td>
<td></td>
</tr>
</tbody>
</table>

or

<table>
<thead>
<tr>
<th></th>
<th>u</th>
<th>v</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Relations and functions

- Relations represent **one to many relationships** between elements in $A$ and $B$.
- Example:

```
   a
  /  \
 /    \n b  ->  1
    /   \
   \    
    2    
      \  
        3
```

- What is the difference between a **relation and a function from $A$ to $B$?**
Relations and functions

- Relations represent **one to many relationships** between elements in A and B.

- Example:

```
   a   1
   b   2
         3
```

- What is the difference between a relation and a function from A to B? A function on sets A,B \( A \rightarrow B \) assigns to each element in the domain set A exactly one element from B. So it is a special relation.

```
   a   1
       2
   b   3
```

Relation on the set

**Definition:** A relation on the set A is a relation from A to itself.

**Example 1:**

- Let \( A = \{1,2,3,4\} \) and \( R_{\text{div}} = \{(a,b)| a \text{ divides } b\} \)
- What does \( R_{\text{div}} \) consist of?
- \( R_{\text{div}} = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\} \)

```
<table>
<thead>
<tr>
<th>R</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
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<tbody>
<tr>
<td>1</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
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<tr>
<td>3</td>
<td></td>
<td></td>
<td>x</td>
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</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td>x</td>
</tr>
</tbody>
</table>
```
Relation on the set

Example:
• Let $A = \{1,2,3,4\}$.
• Define a $R \neq b$ if and only if $a \neq b$.

$R$ = \{(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,4),(4,1),(4,2),(4,3)\}

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>x</td>
<td>x</td>
<td>x</td>
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<tr>
<td>2</td>
<td></td>
<td>x</td>
<td>x</td>
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<td>3</td>
<td>x</td>
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<td>x</td>
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<tr>
<td>4</td>
<td>x</td>
<td>x</td>
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</tbody>
</table>

Binary relations

• **Theorem:** The number of binary relations on a set $A$, where $|A| = n$ is:

$$2^{n^2}$$

• **Proof:**
  • If $|A| = n$ then the cardinality of the Cartesian product $|A \times A| = n^2$.
  • $R$ is a binary relation on $A$ if $R \subseteq A \times A$ (that is, $R$ is a subset of $A \times A$).
  • The number of subsets of a set with $k$ elements : $2^k$
Binary relations

- **Theorem:** The number of binary relations on a set $A$, where $|A| = n$ is:
  \[2^n^2\]

- **Proof:**
  - If $|A| = n$ then the cardinality of the Cartesian product $|A \times A| = n^2$.
  - $R$ is a binary relation on $A$ if $R \subseteq A \times A$ (that is, $R$ is a subset of $A \times A$).
  - The number of subsets of a set with $k$ elements: $2^k$.
  - The number of subsets of $A \times A$ is: $2^{|A \times A|} = 2^{n^2}$.

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Example: Let $A = \{1,2\}$

- What is $A \times A = \{(1,1),(1,2),(2,1),(2,2)\}$

- List of possible relations (subsets of $A \times A$):
  - $\emptyset$  
  - $\{(1,1)\} \quad \{(1,2)\} \quad \{(2,1)\} \quad \{(2,2)\}$  
  - $\{(1,1),(1,2)\} \quad \{(1,1),(2,1)\} \quad \{(1,1),(2,2)\}$  
  - $\{(1,2),(2,1)\} \quad \{(1,2),(2,2)\} \quad \{(2,1),(2,2)\}$  
  - $\{(1,1),(1,2),(2,1)\} \quad \{(1,1),(1,2),(2,2)\}$  
  - $\{(1,1),(2,1),(2,2)\} \quad \{(1,2),(2,1),(2,2)\}$  
  - $\{(1,1),(1,2),(2,1),(2,2)\}$  

- Use formula: $2^4 = 16$
Properties of relations

**Definition (reflexive relation):** A relation $R$ on a set $A$ is called **reflexive** if $(a,a) \in R$ for every element $a \in A$.

**Example 1:**
- Assume relation $R_{\text{div}} = \{(a,b), \text{ if } a \mid b\}$ on $A = \{1,2,3,4\}$
- **Is $R_{\text{div}}$ reflexive?**
  - $R_{\text{div}} = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$
  - **Answer:** Yes. $(1,1), (2,2), (3,3), \text{ and } (4,4) \in A.$

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**Reflexive relation**

**Reflexive relation**
- $R_{\text{div}} = \{(a,b), \text{ if } a \mid b\}$ on $A = \{1,2,3,4\}$
- $R_{\text{div}} = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$

$$
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{array}
$$

- A relation $R$ is reflexive if and only if $MR$ has 1 in every position on its main diagonal.
Properties of relations

**Definition (reflexive relation):** A relation \( R \) on a set \( A \) is called **reflexive** if \((a,a) \in R\) for every element \( a \in A\).

**Example 2:**
- Relation \( R_{\text{fun}} \) on \( A = \{1,2,3,4\} \) defined as:
  - \( R_{\text{fun}} = \{(1,2),(2,2),(3,3)\} \).
- **Is** \( R_{\text{fun}} \) **reflexive?**
- **No.** It is not reflexive since \((1,1) \not\in R_{\text{fun}}\).

Properties of relations

**Definition (irreflexive relation):** A relation \( R \) on a set \( A \) is called **irreflexive** if \((a,a) \not\in R\) for every \( a \in A\).

**Example 1:**
- Assume relation \( R_{\neq} \) on \( A = \{1,2,3,4\} \), such that \( a R_{\neq} b \) if and only if \( a \neq b \).
- **Is** \( R_{\neq} \) **irreflexive?**
- **Yes.** Because \((1,1),(2,2),(3,3), \) and \((4,4) \not\in R_{\neq}\).
Irreflexive relation

An irreflexive relation $R$ on a set $A = \{1,2,3,4\}$, such that $a \not R b$ if and only if $a \neq b$.

- $R = \{(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,4),(4,1),(4,2),(4,3)\}$

$$
\begin{array}{cccc}
0 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 \\
1 & 1 & 1 & 0 \\
\end{array}
$$

- A relation $R$ is irreflexive if and only if $M_R$ has 0 in every position on its main diagonal.

Properties of relations

**Definition (irreflexive relation):** A relation $R$ on a set $A$ is called **irreflexive** if $(a,a) \not \in R$ for every $a \in A$.

**Example 2:**

- $R_{fun}$ on $A = \{1,2,3,4\}$ defined as:
  - $R_{fun} = \{(1,2),(2,2),(3,3)\}$.
- Is $R_{fun}$ irreflexive?
- **Answer:** No. Because $(2,2)$ and $(3,3) \not \in R_{fun}$.
Properties of relations

**Definition (symmetric relation):** A relation R on a set A is called **symmetric** if
\[ \forall a, b \in A \quad (a, b) \in R \rightarrow (b, a) \in R. \]

Example 1:
- \( R_{\text{div}} = \{(a, b), \text{if } a | b\} \) on \( A = \{1, 2, 3, 4\} \)
- Is \( R_{\text{div}} \) symmetric?
- \( R_{\text{div}} = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\} \)
- **Answer:** No. It is not symmetric since \( (1,2) \in R \) but \( (2,1) \notin R \).

Example 2:
- \( R_{\neq} \) on \( A = \{1, 2, 3, 4\} \), such that \( a \ R_{\neq} b \) if and only if \( a \neq b \).
- Is \( R_{\neq} \) symmetric?
- \( R_{\neq} = \{(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,4),(4,1),(4,2),(4,3)\} \)
- **Answer:** Yes. If \( (a,b) \in R_{\neq} \rightarrow (b,a) \in R_{\neq} \)
Symmetric relation

Symmetric relation:
- $R_{\neq}$ on $A=\{1,2,3,4\}$, such that $a \neq b$ if and only if $a \neq b$.
- $R_{\neq} = \{(1,2), (1,3), (1,4), (2,1), (2,3), (2,4), (3,1), (3,2), (3,4), (4,1), (4,2), (4,3)\}$

- $A$ relation $R$ is symmetric if and only if $m_{ij} = m_{ji}$ for all $i,j$.

Properties of relations

**Definition (symmetric relation):** A relation $R$ on a set $A$ is called symmetric if

\[ \forall a, b \in A \quad (a,b) \in R \rightarrow (b,a) \in R. \]

**Example 3:**
- Relation $R_{\text{fun}}$ on $A = \{1,2,3,4\}$ defined as:
  - $R_{\text{fun}} = \{(1,2), (2,2), (3,3)\}$.
- Is $R_{\text{fun}}$ symmetric?
- **Answer:** No. For $(1,2) \in R_{\text{fun}}$ there is no $(2,1) \in R_{\text{fun}}$
Properties of relations

**Definition (anti-symmetric relation):** A relation on a set $A$ is called **anti-symmetric** if

\[
[(a, b) \in R \text{ and } (b, a) \in R] \implies a = b \text{ where } a, b \in A.
\]

**Example 1:**

- $R_{\text{div}} = \{(a, b), \text{ if } a \mid b\}$ on $A = \{1, 2, 3, 4\}$
- $R_{\text{div}} = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$
- **Is $R_{\text{div}}$ anti-symmetric?**
- **Answer:** Yes. There is no $(a, b)$ and $(b, a)$ in $R$ for $a \neq b$.

**Properties of relations**

**Definition (antisymmetric relation):** A relation on a set $A$ is called **antisymmetric** if

\[
[(a, b) \in R \text{ and } (b, a) \in R] \implies a = b \text{ where } a, b \in A.
\]

**Example 2:**

- $R_{\neq}$ on $A = \{1, 2, 3, 4\}$, such that $a R_{\neq} b$ if and only if $a \neq b$.
- **Is $R_{\neq}$ antisymmetric?**
- **$R_{\neq} = \{(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,4),(4,1),(4,2),(4,3)\}$**
- **Answer:** No. It is not anti-symmetric since $(1, 2) \in R$ and $(2, 1) \in R$ but $1 \neq 2$. 
Properties of relations

**Definition (anti-symmetric relation):** A relation on a set A is called **anti-symmetric** if

- \([ (a,b) \in R \text{ and } (b,a) \in R ] \rightarrow a = b \) where \( a, b \in A \).

**Example 3:**
- Relation \( R_{fun} \) on \( A = \{1,2,3,4\} \) defined as:
  - \( R_{fun} = \{(1,2),(2,2),(3,3)\} \).
- **Is** \( R_{fun} \) **anti-symmetric?**
- **Answer:** Yes. It is anti-symmetric

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Antisymmetric relations

**Antisymmetric relation**
- relation \( R_{fun} = \{(1,2),(2,2),(3,3)\} \)

\[
MR_{fun} = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

- A relation is **antisymmetric** if and only if \( m_{ij} = 1 \rightarrow m_{ji} = 0 \) for \( i \neq j \).
Properties of relations

Definition (transitive relation): A relation $R$ on a set $A$ is called transitive if
- $[(a,b) \in R \text{ and } (b,c) \in R] \implies (a,c) \in R$ for all $a, b, c \in A$.

Example 1:
- $R_{\text{div}} = \{(a,b), \text{ if } a \mid b\}$ on $A = \{1,2,3,4\}$
- $R_{\text{div}} = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$
- Is $R_{\text{div}}$ transitive?
- Answer: Yes.