CS 441 Discrete Mathematics for CS

Propositional logic

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Course administrivia

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Course web page:
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Course administrivia

Lectures:

- Monday, Wednesday: 11:00 AM - 12:15 PM
- 5129 SENSQ

Recitations:

- held on Wednesdays in 5313 SENSQ
  - **Section 1:** 1:00 – 1:50 PM
  - **Section 2:** 2:00 – 2:50 PM

Announcement

- **No recitations this week:**
  - **recitations start on January 16**
- **First homework assignment will be out on January 14**

Discrete mathematics

- **Discrete mathematics**
  - the study of discrete mathematical structures and objects (as opposed to continuous objects).
- **Examples of objects** with discrete values are integers, graphs, or statements in logic.
- Discrete mathematics and **computer science**. Concepts from discrete mathematics are useful for describing objects and problems in computer algorithms and programming languages. These have applications in cryptography, automated theorem proving, and software development.

Course syllabus

Tentative topics:

- **Logic and proofs**
- **Sets**
- **Functions**
- **Integers and modular arithmetic**
- **Sequences and summations**
- **Counting**
- **Probability**
- **Relations**

Course syllabus

Tentative topics:

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Logic

Logic:

- defines a formal language for representing knowledge and for making logical inferences
- It helps us to understand how to construct a valid argument

Logic defines:

- Syntax of statements
- The meaning of statements
- The rules of logical inference

Propositional logic

- **The simplest logic**
- **Proposed by George Boole (1815-1864)**

- **Definition:**
  - **A proposition is a statement that is either true or false.**

- **Examples:**
  - Pitt is located in the Oakland section of Pittsburgh.
    - **(T)**

Propositional logic

- **The simplest logic**

- **Definition:**
  - **A proposition is a statement that is either true or false.**

- **Examples:**
  - Pitt is located in the Oakland section of Pittsburgh.
    - **(T)**
  - $5 + 2 = 8.$
    - **(F)**
  - It is raining today.
    - **(either T or F)**

Propositional logic

- **Examples (cont.):**
  - How are you?
    - **a question is not a proposition**
  - x + 5 = 3
    - **since x is not specified, neither true nor false**
  - 2 is a prime number.
    - **(T)**
  - She is very talented.
    - **since she is not specified, neither true nor false**
  - There are other life forms on other planets in the universe.
    - **either T or F**

Composite statements

- More complex propositional statements can be build from elementary statements using **logical connectives.**

- **Logical connectives:**
  - Negation
  - Conjunction
  - Disjunction
  - Exclusive or
  - Implication
  - Biconditional

Negation

- **Defn**: Let p be a proposition. The statement "It is not the case that p." is another proposition, called the **negation of p**. The negation of p is denoted by $\neg$ p and read as "not p."

- **Examples:**
  - It is not the case that Pitt is located in the Oakland section of Pittsburgh.
  - $5 + 2 \neq 8$.
  - 10 is not a prime number.
  - It is not the case that buses stop running at 9:00pm.

Negation

- **Negate the following propositions:**
  - It is raining today.
    - It is not raining today.
  - 2 is a prime number.
    - 2 is not a prime number
  - There are other life forms on other planets in the universe.
    - It is not the case that there are other life forms on other planets in the universe.

Negation

- A **truth table** displays **the relationships between truth values** (T or F) of propositions.

p	¬p
T	F
F	T

Conjunction

- **Definition**: Let p and q be propositions. The proposition "**p and q**" denoted by $p \wedge q$, is true when both p and q are true and is false otherwise. The proposition $p \wedge q$ is called the **conjunction** of p and q.

- **Examples:**
  - Pitt is located in the Oakland section of Pittsburgh and 5 + 2 = 8
  - It is raining today and 2 is a prime number.
  - 2 is a prime number and $5 + 2 \neq 8$.
  - 13 is a perfect square and 9 is a prime.

Disjunction

- **Definition**: Let p and q be propositions. The proposition "**p or q**" denoted by $p \vee q$, is false when both p and q are false and is true otherwise. The proposition $p \vee q$ is called the **disjunction** of p and q.

- **Examples:**
  - Pitt is located in the Oakland section of Pittsburgh or 5 + 2 = 8.
  - It is raining today or 2 is a prime number.
  - 2 is a prime number or $5 + 2 \neq 8$.
  - 13 is a perfect square or 9 is a prime.

Truth tables

- **Conjunction and disjunction**
- Four different combinations of values for p and q

p	q	$p \wedge q$	$p \vee q$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	F

- NB: $p \vee q$ (the or is used inclusively, i.e., $p \vee q$ is true when either p or q or both are true).

Exclusive or

- **Definition:** Let p and q be propositions. The proposition "p exclusive or q" denoted by $p \oplus q$, is true when exactly one of p and q is true and it is false otherwise.

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Implication

- **Definition**: Let p and q be propositions. The proposition "**p implies q**" denoted by $p \rightarrow q$ is called implication. It is false when p is true and q is false and is true otherwise.

- In $p \rightarrow q$, p is called the **hypothesis** and q is called the **conclusion**.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Implication

- $p \rightarrow q$ is read in a variety of equivalent ways:
    - if p then q
    - p only if q
    - p is sufficient for q
    - q whenever p

- **Examples:**
  - if Steelers win the Super Bowl in 2013 then 2 is a prime.
    - What is the truth value ?
  - if today is wednesday then 2 * 3 = 8.
    - What is the truth value ?

Implication

- $p \rightarrow q$ is read in a variety of equivalent ways:
    - if p then q
    - p only if q
    - p is sufficient for q
    - q whenever p

- **Examples:**
  - if Steelers win the Super Bowl in 2013 then 2 is a prime.
    - T
  - if today is tuesday then 2 * 3 = 8.
    - F

Implication

- The **converse of $p \rightarrow q$** is $q \rightarrow p$.
- The **contrapositive of $p \rightarrow q$** is $\neg q \rightarrow \neg p$
- The **inverse** of **$p \rightarrow q$** is $\neg p \rightarrow \neg q$

- **Examples:**
  - If it snows, the traffic moves slowly.
  - p: it snows   q: traffic moves slowly.
  - $\mathbf{p \rightarrow q}$
  - **The converse:**
    If the traffic moves slowly then it snows.
    - $\mathbf{q \rightarrow p}$

Implication

- The **contrapositive of $p \rightarrow q$** is $\neg q \rightarrow \neg p$
- The **inverse** of **$p \rightarrow q$** is $\neg p \rightarrow \neg q$

- **Examples:**
  - If it snows, the traffic moves slowly.
  - **The contrapositive:**
    - If the traffic does not move slowly then it does not snow.
    - $\neg q \rightarrow \neg p$
  - **The inverse:**
    - If it does not snow the traffic moves quickly.
    - $\neg p \rightarrow \neg q$

Biconditional

- **Definition**: Let p and q be propositions. The biconditional $p \leftrightarrow q$ (**read p if and only if q**), is true when p and q have the same truth values and is false otherwise.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

- **Note:** two truth values always agree.

Constructing the truth table

- **Example: Construct a truth table for**
  $(p \rightarrow q) \wedge (\neg p \leftrightarrow q)$
- Simpler if we decompose the sentence to elementary and intermediate propositions

p	q	$\neg p$	$p \rightarrow q$	$\neg p \leftrightarrow q$	$(p\rightarrow q)\wedge(\neg p\leftrightarrow q)$
T	T				
T	F				
F	T				
F	F				

Constructing the truth table

- Example: Construct the truth table for
  $(p \rightarrow q) \wedge (\neg p \leftrightarrow q)$

Rows: all possible combinations of values for elementary propositions: 2^n values

p	q	¬p			$(p \rightarrow q) \wedge (\neg p \leftrightarrow q)$
T	T				
T	F				
F	T				
F	F				

Constructing the truth table

- Example: Construct the truth table for
  $(p \rightarrow q) \wedge (\neg p \leftrightarrow q)$

Typically the target (unknown) compound proposition and its values

p	q	¬p	$p \rightarrow q$	$\neg p \leftrightarrow q$	$(p \rightarrow q) \wedge (\neg p \leftrightarrow q)$
T	T				
T	F				
F	T				
F	F				

Auxiliary compound propositions and their values

Constructing the truth table

- **Examples: Construct a truth table for**

  $(p \rightarrow q) \wedge (\neg p \leftrightarrow q)$

  Simpler if we decompose the sentence to elementary and intermediate propositions

p	q	$\neg p$	$p \rightarrow q$	$\neg p \leftrightarrow q$	$(p\rightarrow q)\wedge(\neg p\leftrightarrow q)$
T	T	F	T	F	F
T	F	F	F	T	F
F	T	T	T	T	T
F	F	T	T	F	F

Translation

Logic helps us to define the meaning of statements:

- Mathematical or English statements.

Question: **How to translate an English sentence to the logic?**

Assume a sentence:

- If you are older than 13 or you are with your parents then you can attend a PG-13 movie.

- The whole sentence is a proposition. It is **True.**

- But this is not the best. We want to parse the sentence to elementary statements that are combined with connectives.

Translation

If you are older than 13 or you are with your parents then you can attend a PG-13 movie.

Parse:

- **If (you are older than 13 or you are with your parents) then (you can attend a PG-13 movie)**
  - A= you are older than 13
  - B= you are with your parents
  - C=you can attend a PG-13 movie
- **Translation: $A \vee B \rightarrow C$**

- **But why do we want to do it this way?**

Translation

Assume we know:

If you are older than 13 or you are with your parents then you can attend a PG-13 movie. You are older than 13.

Parse:

- **If (** you are older than 13 **or** you are with your parents **) then (** you can attend a PG-13 movie**)** . (**Y**ou are older than 13).
  - A= you are older than 13
  - B= you are with your parents
  - C=you can attend a PG-13 movie
- **Translation: $(A \vee B \rightarrow C)$, A**

- **<u>With the help of the logic we can infer the following statement (proposition):</u>**
  - **Y**ou can attend a PG-13 movie