CS 441 Discrete Mathematics for CS
Lecture 19

Probabilities III

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Probabilities

- **Experiment:**
  - a procedure that yields one of the possible outcomes
- **Sample space:** a set of possible outcomes
- **Event:** a subset of possible outcomes (E is a subset of S)

- **Assuming the outcomes are equally likely, the probability of an event E, defined by a subset of outcomes from the sample space S is**
  - **P(E)= |E| / |S|**

- The cardinality of the subset divided by the cardinality of the sample space.

Probabilities

Three axioms of the probability theory:

(1) Probability of a discrete outcome s is:

- 0 < = P(s) <= 1

(2) Sum of probabilities of all disjoint outcomes is = 1

(3) For any two events E1 and E2 holds:

P(E1 U E2) = P(E1) + P(E2) – P(E1 and E2)

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Probability distribution

Definition: A function **p: S → [0,1]** satisfying the three axioms of probability is called **a probability distribution**

Example: a biased coin

- Probability of head 0.6, probability of a tail 0.4
- **Probability distribution:**
  - Head → 0.6 The sum of the probabilities sums to 1
  - Tail → 0.4

Note: a uniform distribution is a special distribution that assigns equal probabilities to each outcome.

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Conditional probability

Definition: Let E and F be two events such that P(F) > 0. The **conditional probability** of E given F

- $P(E|F) = P(E \wedge F) / P(F)$

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Conditional probability

Corrolary: Let E and F be two events such that P(F) > 0. Then:

- $P(E \wedge F) = P(E|F) P(F)$

Example:

- Assume the probability of getting a flu is 0.2
- Assume the probability of having a high fever given the flu: 0.9

What is the probability of getting a flu with fever?

P(flu and fever) = P(fever|flu)P(flu) = 0.9*0.2 = 0.18

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Bayes theorem

Definition: Let E and F be two events such that P(F) > 0. Then:

- $P(E| F) = P(F| E)P(E) / P(F)$

Proof:

$P(E| F) = P(E \wedge F) / P(F)$

$= P(F| E) P(E) / P(F)$

Idea: Simply switch the conditioning events.

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Bayes theorem

Definition: Let E and F be two events such that P(F) > 0. Then:

- $P(E|F) = P(F|E)P(E) / P(F)$

Example:

- Assume the probability of getting a flu is 0.2
- Assume the probability of getting a fever is 0.3
- Assume the probability of having a high fever given the flu: 0.9
- What is the probability of having a flu given the fever?
- P(flu | fever) = P(fever|flu) P(flu) / P(fever) =

  = 0.9 x 0.2/0.3 = 0.18/0.6 = 0.3

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Independence

Definition: The two events E and F are said to be **independent** if:

- $P(E \wedge F) = P(E)P(F)$

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Random variables

- **Definition: A random variable** is a function from the sample space of an experiment to the set of real numbers f: S → R. A random variable assigns a number to each possible outcome.
- **The distribution of a random variable X on the sample space S** is a set of pairs (r p(X=r)) for all r in S where r is the number and p(X=r) is the probability that X takes a value r.

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Random variables

Example:

Let S be the outcomes of a two-dice roll

Let random variable X denotes the sum of outcomes

(1,1) → 2

(1,2) and (2,1) → 3

(1,3), (3,1) and (2,2) → 4

…

Distribution of X:

- 2 → 1/36,
- 3 → 2/36,
- 4 → 3/36 …
- 12 → 1/36

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Probabilities

- **Assume a repeated coin flip**
- P(head) =0.6 and the probability of a tail is 0.4. Each coin flip is independent of the previous.
- What is the probability of seeing:
  - HHHHH - 5 heads in a row
- **P(HHHHH) = 0.6^5 =**
  - Assume the outcome is HHTTT
- **P(HHTTT)= $0.6*0.6*0.4^{3}$= $0.6^2*0.4^3$**
  - Assume the outcome is TTHHT
- **P(TTHHT)=$0.4^{2}*0.6^2*0.4 = 0.6^2*0.4^3$**
- What is the probability of seeing three tails and two heads?
- The number of two-head-three tail combinations = **C(5,2)**
- P(two-heads-three tails) = **C(5,2) $*0.6^2*0.4^3$**

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Probabilities

- **Assume a variant of a repeated coin flip problem**
- The space of possible outcomes is the count of occurrences of heads in 5 coin flips. For example:
- TTTTT yields outcome 0
- HTTTT or TTHTT yields 1
- HTHHT yields 3 …

- What is the probability of an outcome 0?
- P(outcome=0) = $\mathbf{0.6^0 * 0.4^5}$
- P(outcome=1) = **C(5,1)** $\mathbf{0.6^1 * 0.4^4}$
- P(outcome =2) = **C(5,2)** $\mathbf{0.6^2 * 0.4^3}$
- P(outcome =3) = **C(5,3)** $\mathbf{0.6^3 * 0.4^2}$
- …

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Expected value and variance

Definition: The expected value of the random variable X(s) on the sample space is equal to:

$$E(X) = \sum_{s \in S} p(s) X(s)$$

Example: roll of a dice

- Outcomes: 1 2 3 4 5 6
- Expected value:

E(X) = 1*1/6 + 2*1/6+3*1/6 + 4*1/6 + 5*1/6 +6* 1/6 = 7/2=3.5

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Expected value

Example:

Flip a fair coin 3 times. The outcome of the trial X is the number of heads. What is the expected value of the trial?

Answer:

Possible outcomes:

= {HHH	**HHT**	**HTH**	**THH**	**HTT**	**THT**	**TTH**	**TTT}**
3	**2**	**2**	**2**	**1**	**1**	**1**	**0**

E(X) = 1/8 (3 + 3*2 +3*1 +0) = 12/8= 3/2

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Expected value

- **Theorem:** If Xi i=1,2,3, n with n being a positive integer, define random variables on S, and a and b are real numbers then:
  - E(X1+X2+ ... Xn) = E(X1)+E(X2) + ... E(Xn)
  - E(aX+b) = aE(X) +b

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Expected value

Example:

- Roll a pair of dices. What is the expected value of the sum of outcomes?
- **Approach 1:**
- Outcomes: (1,1) (1,2) (1,3) .... (6,1)... (6,6)

  2 3 4 7 12

Expected value: 1/36 (2*1 + ...) = 7

- **Approach 2 (theorem):**
- E(X1+X2) = E(X1) +E(X2)
- E(X1) =7/2 and E(X2) = 7/2
- E(X1+X2) = 7

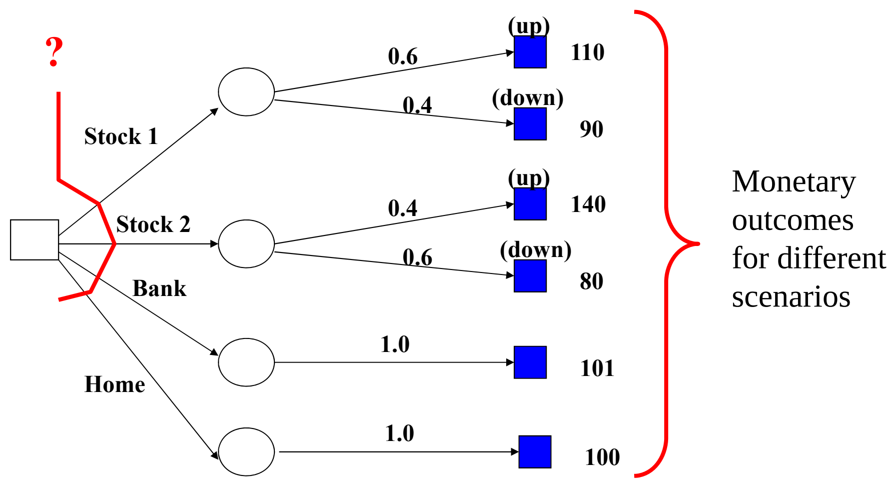
Expected value

Investment problem:

- You have 100 dollars and can invest into a stock. The returns are volatile and you may get either $120 with probability of 0.4, or $90 with probability 0.6.
- **What is the expected value of your investment?**
- E(X) = 0.4*120+0.6*90=48+54=102
- **Is it OK to invest?**

Decision making example

We need to make a choice whether to invest in Stock 1 or 2, put money into bank or keep them at home.



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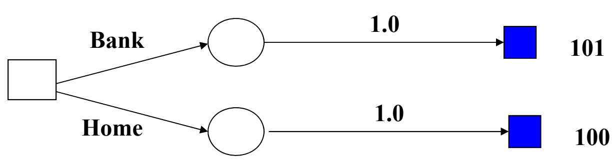
Decision making example.

Assume the simplified problem with the Bank and Home choices only.

The result is guaranteed – the outcome is deterministic



What is the rational choice assuming our goal is to make money?

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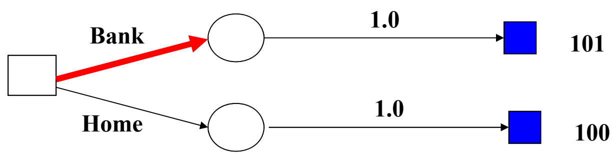
Decision making. Deterministic outcome.

Assume the simplified problem with the Bank and Home choices only.

These choices are deterministic.



Our goal is to make money. What is the rational choice?

Answer: Put money into the bank. The choice is always strictly better in terms of the outcome

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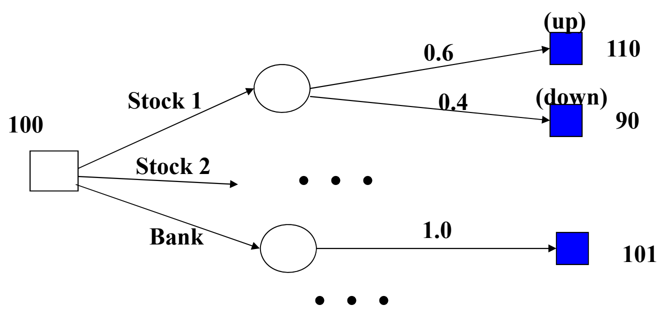
Decision making

- **How to quantify the goodness of the stochastic outcome?**
  We want to compare it to deterministic and other stochastic outcomes.



Idea: Use the expected value of the outcome

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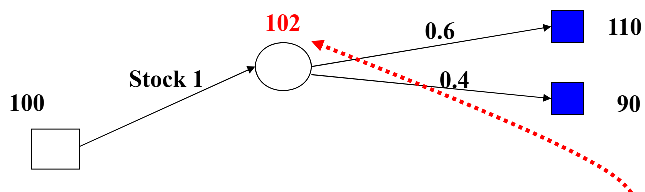
Expected value

- **Expected value** summarizes all stochastic outcomes into a single quantity
  - Expected value for the outcome of the Stock 1 option is:



$$0.6 \times 110 + 0.4 \times 90 = 66 + 36 = 102$$

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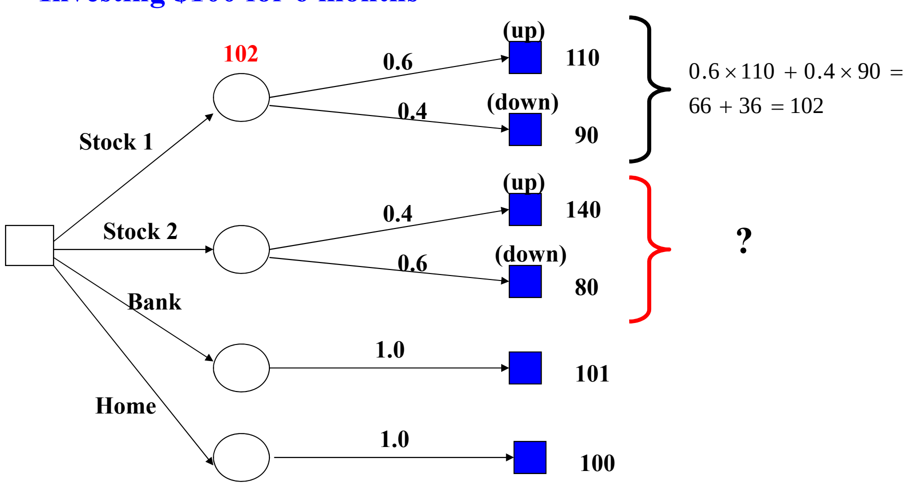
Expected values

Investing $100 for 6 months



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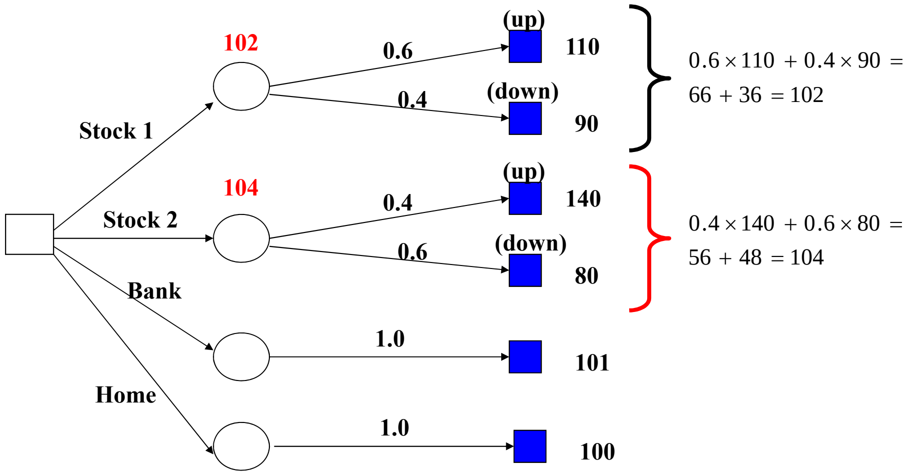
Expected values

Investing \$100 for 6 months



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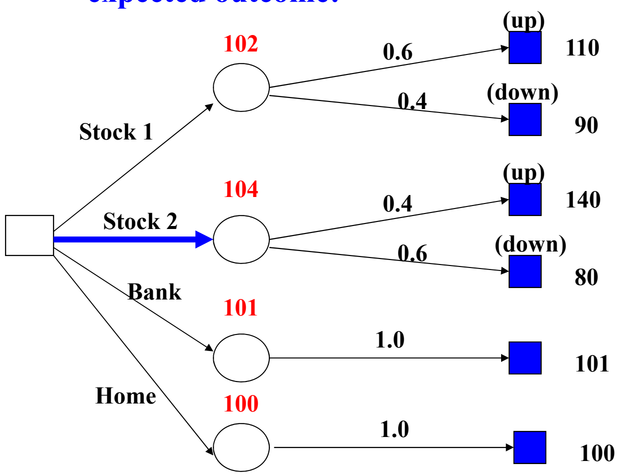
Selection based on expected values

The optimal action is the option that maximizes the expected outcome:



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