CS 441 Discrete Mathematics for CS
Lecture 17

Counting

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Course administration

- **Homework 7 : Due today**
- **Homework 8: Due on Monday April 1, 2013**
- **Midterm exam 2**
  - **Monday, March 25, 2013**
  - **Covers only the material after midterm 1**
    - **Integers (Primes, Division, Congruencies)**
    - **Sequences and Summations**
    - **Inductive proofs and Recursion**
    - **Counting**

Course web page:

http://www.cs.pitt.edu/~milos/courses/cs441/

Permutations

A permutation of a set of distinct objects is an **ordered arrangement** of the objects. Since the objects are distinct, they cannot be selected more than once. Furthermore, the order of the arrangement matters.

Example:

- Assume we have a set S with n elements. S={a,b,c}.
- **Permutations of S:**
- **a b c   a c b   b a c   b c a   c a b   c b a**

The number of permutations of n elements is:

P(n,n) = n!

k-permutations

- ***k*-permutation** is an ordered arrangement of *k* elements of a set.

Example:

- Assume we have a set S with n elements. S={a,b,c}.
- **2 permutations of S:**
- **ab ba ac ca bc cb**

- The number of *k*-permutations of a set with *n* distinct elements is:

P(n,k) = n(n-1)(n-2)…(n-k+1)= n!/(n-k)!

Combinations

A *k*-combination of elements of a set is an **unordered** selection of k elements from the set. Thus, a k-combination is simply a subset of the set with k elements.

Example:

- 2-combinations of the set {a,b,c}

  a b a c b c

  ↓

  a b covers two of the 2-permutations: **a b** and **b a**

Combinations

Theorem: The number of k-combinations of a set with n distinct elements, where n is a positive integer and k is an integer with $0 \leq k \leq n$ is

$$C(n,k) = \frac{n!}{(n-k)!\,k!}$$

Binomial coefficients

- The number of k-combinations out of n elements C(n,k) is often denoted as:

$$\binom{n}{k}$$

and reads **n choose k**. The number is also called **a binomial coefficient.**

- Binomial coefficients occur as coefficients in the expansion of powers of binomial expressions such as

$$(a+b)^n$$

- **Definition:** a binomial expression is the sum of two terms *(a+b).*

Binomial coefficients

Example:

- Expansion of the binomial expression $(a+b)^3$.

$$(a+b)^3 =$$
$$(a+b)(a+b)(a+b) =$$
$$(a^2+2ab+b^2)(a+b) =$$
$$a^3+2a^2b+ab^2+a^2b+2ab^2+b^3 =$$
$$1a^3+3a^2b+3ab^2+1b^3$$

1 **3** **3** **1** ← Binomial coefficients

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

Binomial coefficients

Binomial theorem: Let a and b be variables and n be a nonnegative integer. Then:

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i$$

$$= \binom{n}{0} a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \dots + \binom{n}{n-1} a^1 b^{n-1} + \binom{n}{n} b^n$$

Binomial coefficients

Question: We have binomial coefficients for expressions with the power n. Are binomial coefficients for powers of n-1 or n+1 in any way related to coefficients for n ?

- **The answer is yes.**

Theorem:

- Let n and k be two positive integers with . Then it holds:

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

Pascal traingle

Drawing the binomial coefficients for different powers in increasing order gives a Pascal triangle:

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

	powers
1	
1 1	
1 2 1	**2**
1 3 3 1	**3**
1 4 6 4 1	**4**
1 5 10 10 5 1	**5**
1 6 15 20 15 6 1	**6**
…	**7**

Permutations with repetitions

Assume we want count different ordered collections of objects such that we can pick the object from a set multiple times: that is, an object can be on the position 1, 2, 3, etc

Example:

- 26 letters of alphabet. How many different strings of length k are there?

Answer:

- 26^k

Theorem: The number of k-permutations of a set of n objects with repetition is $\mathbf{n^k}$.

Combinations with repetitions

Example:

- Pick four pieces of a fruit from three bowls with apples, pears and oranges. How many possible combinations are possible? List /count all of them?

Answer:

- **Star and bar approach**
- Apples Pears Oranges
- 3 bowls separated by | |
- Choice 2 apples and 2 pears represented as: ** | ** |
- Choice of 1 apple and 3 oranges: * | | ***
- **Count:** How many different ways of arranging (3-1)=2 bars and 4 stars are there?
- Total number of positions: 2+4=6

Combinations with repetitions

- **Theorem:** The number of ways to pick n elements from k different groups is:

$$\binom{n-1+k}{n}$$

- (n+k-1) positions
- n- stars
- **Count:** the number of ways to select the positions of 4 stars.

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Probabilities

Probability

Discrete probability theory

- Dates back to 17th century.
- It was used to compute the odds of seeing some outcomes: e.g. in games, races etc.
- Odds are related to counting when the outcomes are equally likely.

Example: Coin flip

- Assume 2 outcomes (head and tail) and each of them is equally likely
- Odds: 50%, 50%
- the probability of seeing:
  - a head is 0.5
  - a tail is 0.5

Probability

Probability is related to relative **counts** of target outcomes with respect to all outcomes

P = Number of target outcomes / Total number of outcomes

Example: Coin flip

- the probability of seeing:
  - a head is 0.5
  - a tail is 0.5
- Probability of all (disjoint) events is 1
- **P(any outcome) = P(head) + P(tail) =1**

Probability

Example: roll of a dice

- 6 different outcomes. Each of them is equally likely
- Probability of each outcome is:
- 1/6
- How did we get the number?
- Assuming each outcome is equally likely and we have 6 outcomes, then the probability of an outcome is:

  1/6

Probability of aggregate outcomes

Example: roll of a dice

- Roll of the dice is odd or even. All outcomes are equally likely.
- Probability = number of outcomes when odd/ total number of outcomes.
- P(odd) = ?
- P(even)=?

Probability of aggregate outcomes

Example: roll of a dice

- Roll of the dice is odd or even. All outcomes are equally likely.
- Probability = number of outcomes when odd/ total number of outcomes.

Solution 1: all outcomes are equally likely and = 1/6

Odd numbers: 1,3,5 Even numbers: 2,4,6

- P(odd) = 1/6+1/6+1/6 = 3/6 = 1/2
- P(even) = 3/6 = 1/2

Solution 2:

Odd numbers are equally likely as even numbers (2 outcomes)

P(odd) =1/2 and P(even) = 1/2

Probabilities

- **Experiment:** a procedure that yields one of the possible outcomes
- **Sample space:** a set of possible outcomes
- **Event:** a subset of possible outcomes (E is a subset of S)
- **Assuming the outcomes are equally likely, the probability of an event E, defined by a subset of outcomes from the sample space S is**
  - **P(Event)= |E| / |S|**

- The cardinality of the subset divided by the cardinality of the sample space.

Probabilities

Example 1:

- A box with 4 red balls and 6 blue balls. What is the probability that we pull the red ball out.

- **P(E) =6/10 =0.6**

Probabilities

Example 2:

- roll of two dices.
- What is the probability that the outcome is 7.
- Possible outcomes:
- (1,6) (2,6) …(6,1),…(6,6)   total: 36

- Outcomes leading to
- (1,6) (2,5) …(6,1)   total: 6
- P(sum=7)= 6/36 = 1/6

Probabilities

More complex:

- Odd of winning a lottery: 6 numbers out of 40.
- Total number of outcomes:
    - C(40,6) = …
- Probability of winning: ?
    - P(E) =1/C(40,6)= 34! 6! / 40! = 1/3,838,380