CS 441 Discrete Mathematics for CS
Lecture 16

Counting

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Counting

- Assume we have a set of **objects with certain properties**
- **Counting** is used to determine **the number of these objects**

Examples:

- Number of available phone numbers with 7 digits in the local calling area
- Number of possible match starters (football, basketball) given the number of team members and their positions

Basic counting rules

- Counting problems may be hard, and easy solutions are not obvious
- **Approach:**
  - **simplify the solution by decomposing the problem**
- **Two basic decomposition rules:**
  - **Product rule**
    - A count decomposes into a sequence of dependent counts ("each element in the first count is associated with all elements of the second count")
  - **Sum rule**
    - A count decomposes into a set of independent counts ("elements of counts are alternatives")

Inclusion-Exclusion principle

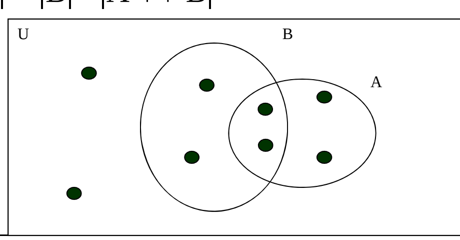
Used in counts where the decomposition yields two count tasks with overlapping elements

- If we used the sum rule some elements would be counted twice

Inclusion-exclusion principle: uses a sum rule and then corrects for the overlapping elements.

We used the principle for the cardinality of the set union.

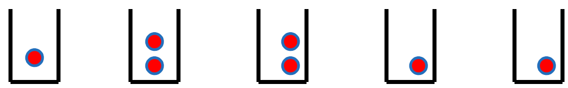
- $|A \cup B| = |A| + |B| - |A \cap B|$



Pigeonhole principle

- Assume you have a set of objects and a set of bins used to store objects.
- **The pigeonhole principle** states that if there are more objects than bins then there is at least one bin with more than one object.

- **Example:** 7 balls and 5 bins to store them
- At least one bin with more than 1 ball exists.



Generalized pigeonhole principle

Theorem. If N objects are placed into k bins then there is at least one bin containing at least $\lceil N / k \rceil$ objects.

Example. Assume 100 people. Can you tell something about the number of people born in the same month.

- Yes. There exists a month in which at least $\lceil 100 / 12 \rceil = \lceil 8.3 \rceil = 9$ people were born.

Permutations

A permutation of a set of distinct objects is an **ordered arrangement** of the objects. Since the objects are distinct, they cannot be selected more than once. Furthermore, the order of the arrangement matters.

Example:

- Assume we have a set S with n elements. S={a,b,c}.
- **Permutations of S:**
- **a b c   a c b   b a c   b c a   c a b   c b a**

Number of permutations

- Assume we have a set S with n elements. S={a_1 a_2 … a_n}.
- **Question:** How many different permutations are there?

- In how many different ways we can choose the first element of the permutation? **n** (**either** a_1 or a_2 … or a_n)
- Assume we picked a_2.
- In how many different ways we can choose the remaining elements? **n-1** (**either** a_1 or a_3 … or a_n **but not** a_2**)**
- **Assume** we picked $a_{j.}$
- In how many different ways we can choose the remaining elements? **n-2** (**either** a_1 or a_3 … or a_n **but not** a_2 **and not** a_j)

  P(n,n) = n.(n-1)(n-2)…1 = n!

Permutations

Example 1.

- How many permutations of letters {a,b,c} are there?
- Number of permutations is:

  $P(n,n) = P(3,3) = 3! = 6$
- Verify:

  abc acb bac bca cab cba

Permutations

Example 2

- How many permutations of letters A B C D E F G H contain a substring ABC.

Idea: consider ABC as one element and D,E,F,G,H as other 5 elements for the total of 6 elements.

Then we need to count the number of permutation of these elements.

$6! = 720$

k-permutations

- ***k*-permutation** is an ordered arrangement of *k* elements of a set.
- The number of *k*-permutations of a set with *n* distinct elements is:

P(n,k) = n(n-1)(n-2)…(n-k+1)= n!/(n-k)!

k-permutations

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Explanation:

- Assume we have a set S with n elements. S={a_1 a_2 … a_n}.
- The 1st element of the *k*-permutation may be any of the *n* elements in the set.
- The 2nd element of the *k*-permutation may be any of the *n-1* remaining elements of the set.
- And so on. For last element of the *k*-permutation, there are *n-k+1* elements remaining to choose from.

k-permutations

Example:

The 2-permutations of set $\{a,b,c\}$ are:

ab, ac, ba, bc, ca, cb.

The number of 2-permutations of this 3-element set is

$P(n,k) = P(3,2) = 3\ (3\text{-}2+1) = 6.$

k-permutations

Example:

Suppose that there are eight runners in a race. The winner receives a gold medal, the second-place finisher receives a silver medal, and the third-place finisher receives a bronze medal. How many different ways are there to award these medals, if all possible outcomes of the race can occur?

Answer:

note that the runners are distinct and that the medals are ordered. The solution is P(8,3) = 8 * 7 * 6 = 8! / (8-3)! = 336.

Combinations

A *k*-combination of elements of a set is an **unordered** selection of k elements from the set. Thus, a k-combination is simply a subset of the set with k elements.

Example:

- 2-combinations of the set {a,b,c}

  a b   a c   b c

  ↓

  a b   covers 2-permutations: **a b** and **b a**

Combinations

Theorem: The number of k-combinations of a set with n distinct elements, where n is a positive integer and k is an integer with $0 \leq k \leq n$ is

$$C(n,k) = \frac{n!}{(n-k)!\,k!}$$

Combinations

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$$C(n,k) = \frac{n!}{(n-k)!\,k!}$$

Proof: The k-permutations of the set can be obtained by first forming the $C(n,k)$ k-combinations of the set, and then ordering the elements in each k-combination, which can be done in $P(k,k)$ ways. Consequently,

$P(n,k) = C(n,k) * P(k,k)$.

This implies that

$C(n,k) = P(n,k) / P(k,k) = P(n,k) / k! = n! / (k! (n-k)!)$

Combinations

Proof (intuition). Assume elements a set {A1, A2, A3, A4, A5}. All 3-combinations of elements are:

- A1 A2 A3
- A1 A2 A4
- A1 A2 A5
- A1 A3 A4
- A1 A3 A5
- A1 A4 A5
- A2 A3 A4
- A2 A3 A5
- A2 A4 A5
- A3 A4 A5
- **Total of 10.**

Combinations

Intuition (example): Assume elements A1, A2, A3, A4 and A5 in the set. All 3-combinations of elements are:

- A1 A2 A3
- A1 A2 A4
- A1 A2 A5
- A1 A3 A4
- A1 A3 A5
- A1 A4 A5
- A2 A3 A4
- A2 A3 A5
- A2 A4 A5
- A3 A4 A5
- **Total of 10.**

Each combination cover many 3-permutations

A1 A2 A3
A1 A3 A2
A2 A1 A3
A2 A3 A1
A3 A1 A2
A3 A2 A1

Combinations

Intuition (example): Assume elements A1, A2, A3, A4 and A5 in the set. All 3-combinations of elements are:

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- A1 A3 A5
- A1 A4 A5
- A2 A3 A4
- A2 A3 A5
- A2 A4 A5
- A3 A4 A5
- **Total of 10.**

Each 3-combination covers many 3-permutations

A1 A2 A3
A1 A3 A2
A2 A1 A3
A2 A3 A1
A3 A1 A2
A3 A2 A1

So: $P(5,3) = C(5,3)\, P(3,3)$

and: $C(5,3) = P(5,3)/P(3,3)$

Combinations

Intuition (example): Assume elements A1, A2, A3, A4 and A5 in the set. All 3-combinations of elements are:

- A1 A2 A3
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- **Total of 10.**

Each 3-combination covers many 3-permutations

A1 A2 A3
A1 A3 A2
A2 A1 A3
A2 A3 A1
A3 A1 A2
A3 A2 A1

So: P(5,3) = C(5,3) P(3,3)

Then: C(5,3) = P(5,3)/P(3,3)

= 5! / (2! 3!)= 10

Combinations

Example:

- We need to create a team of 5 player for the competion out of 10 team members. How many different teams is it possible to create?

Answer:

- When creating a team we do not care about the order in which players were picked. It is is important that the player is in. Because of that we need to consider unordered sets of combinations.
- C(10,5) = 10!/(10-5)!5! = (10.9.8.7.6) / (5 4 3 2 1)

  = 2.3.2.7.3= 6.14.3= 6.42= **252**

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Combinations

Corrolary:

- **C(n,k) = C(n,n-k)**

Proof.

- C(n,k) = n! / (n-k)! k!

  = n!/ (n-k)! (n – (n-k))!

  = C(n,n-k)

Binomial coefficients

- The number of k-combinations out of n elements C(n,k) is often denoted as:

$$\binom{n}{k}$$

  and reads **n choose k**. The number is also called **a binomial coefficient.**
- Binomial coefficients occur as coefficients in the expansion of powers of binomial expressions such as

$$(a+b)^n$$

- **Definition:** a binomial expression is the sum of two terms *(a+b).*

Binomial coefficients

Example:

- Expansion of the binomial expression $(a+b)^3$.

$$(a+b)^3 =$$
$$(a+b)(a+b)(a+b) =$$
$$(a^2+2ab+b^2)(a+b) =$$
$$a^3+2a^2b+ab^2+a^2b+2ab^2+b^3 =$$
$$1a^3+3a^2b+3ab^2+1b^3$$

1 **3** **3** **1** ⇐ Binomial coefficients

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

Binomial coefficients

Binomial theorem: Let a and b be variables and n be a nonnegative integer. Then:

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i$$
$$= \binom{n}{0} a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \dots + \binom{n}{n-1} a^1 b^{n-1} + \binom{n}{n} b^n$$

Binomial coefficients

Binomial theorem: Let a and b be variables and n be a nonnegative integer. Then:

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i$$

- **Proof.** The products after the expansion include terms $a^{(n-i)} b^i$ for all i=0,1, …n. To obtain the number of such coefficients note that we have to choose exactly (n-i) a(s) out of the product of n binomial expressions.

(n-i) picks

$$(a+b)^n = \underbrace{(a+b)(a+b)(a+b)\ldots(a+b)}_{n}$$

- The number of ways we pull a(s) out of the product is given as:

Binomial coefficients

Binomial theorem: Let a and b be variables and n be a nonnegative integer. Then:

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i$$

Proof. The products after the expansion include terms $a^{(n-i)} b^i$ for all i=0,1, …n. To obtain the number of such coefficients note that we have to choose exactly (n-i) a(s) out of the product of n binomial expressions.

(n-i) picks

$$(a+b)^n = \underbrace{(a+b)(a+b)(a+b)\ldots(a+b)}_{n}$$

The number of ways we pull a(s) out is: $\binom{n}{n-i} = \binom{n}{i}$

Binomial coefficients

Corrolary: Let n be a nonnegative integer. Then:

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n$$

Proof:

- Assume a set with n elements:
- C(n,0) = number of subsets of size 0.
- C(n,i) = the number of subsets of size i.
- C(n,n) = the number of subsets of size n.
- The sum of these numbers must give me the number of all subsets of the set n.
- We know it is 2^n so the result follows.

Binomial coefficients

Corrolary:

- Let n be a nonnegative integer. Then:

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} = 0$$

Proof:

$$\sum_{i=0}^{n} \binom{n}{i} (-1)^i = \sum_{i=0}^{n} \binom{n}{i} (-1)^i 1^{n-i} = ((-1)+1)^n = 0^n = 0$$

Binomial coefficients

Example:

- Show that $\sum_{i=0}^{n} \binom{n}{i} 2^i = 3^n$

- **Answer:**

$$\sum_{i=0}^{n} \binom{n}{i} 2^i = \sum_{i=0}^{n} \binom{n}{i} (2)^i 1^{n-i} = (2+1)^n = 3^n$$

Binomial coefficients

Question: We have binomial coefficients for expressions with the power n. Are binomial coefficients for powers of n-1 or n+1 in any way related to coefficients for n ?

- **The answer is yes.**

Theorem:

- Let n and k be two positive integers with . Then it holds:

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

Pascal traingle

Drawing the binomial coefficients for different powers in increasing order gives a Pascal triangle:

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

	powers
1	
1 1	
1 2 1	**2**
1 3 3 1	**3**
1 4 6 4 1	**4**
1 5 10 10 5 1	**5**
1 6 15 20 15 6 1	**6**
…	**7**