CS 441 Discrete Mathematics for CS
Lecture 9

# Inverse functions and composition. Sequences.

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## Course administration

**Midterm:**

- **Tuesday, October 6, 2009**
- **Closed book, in-class**
- **Covers Chapters 1 and 2.1-2.3 of the textbook**

**Homework 4 is out**

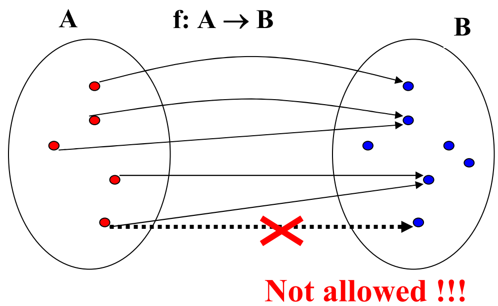
- **Due on Thursday, October 1, 2006**

**Course web page:**

http://www.cs.pitt.edu/~milos/courses/cs441/

# Functions

- **Definition**: Let A and B be two sets. A **function from A to B**, denoted **f : A → B** , is an assignment of exactly one element of B to each element of A. We write f(a) = b to denote the assignment of b to an element a of A by the function f.

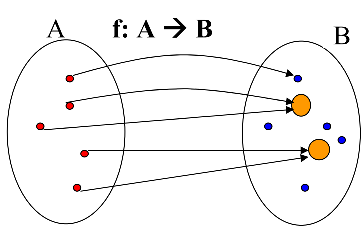




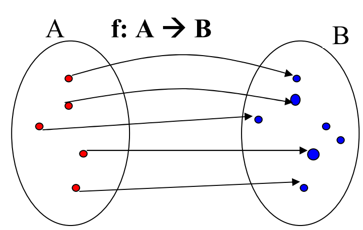

# Injective function

**Definition:** A function f is said to be **one-to-one, or injective**, if and only if f(x) = f(y) implies x = y for all x, y in the domain of f. A function is said to be an **injection if it is one-to-one**.

**Alternate:** A function is one-to-one if and only if f(x) ≠ f(y), whenever x ≠ y. This is the contrapositive of the definition.




**Not injective function**

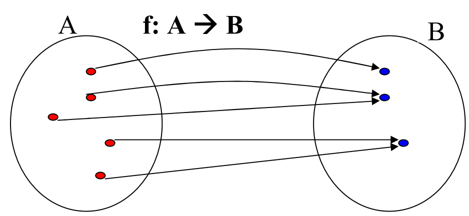



**Injective function**

## Surjective function

**Definition**: A function f from A to B is called **onto**, or **surjective**, if and only if for every b ∈ B there is an element a ∈ A such that f(a) = b.

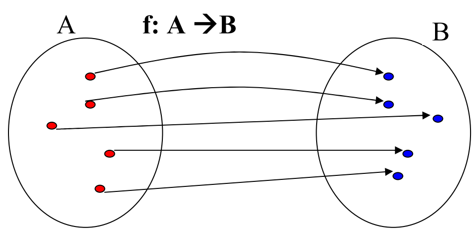
**Alternative:** all co-domain elements are covered






## Bijective functions

**Definition:** A function f is called **a bijection** if it is **both one-to-one and onto**.



## Bijective functions

- Let f be a function from a set A to itself, that is

  f: A → A

**Assume**

- **A is finite** and **f is one-to-one (injective)**
- Is f an onto function (surjection)?
- **Yes.** Every element points to exactly one element. Injection assures they are different. So we have |A| different elements A points to. Since f:A → A the co-domain is covered thus the function is also a surjection (and bijection)
- **A is finite and f is an onto function**
- Is the function one-to-one?



## Bijective functions

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- **Yes.** Every element points to exactly one element. Injection assures they are different. So we have |A| different elements A points to. Since f: A → A the co-domain is covered thus the function is also a surjection (and bijection)
- **A is finite and f is an onto function**
- Is the function one-to-one?
- **Yes.** Every element maps to exactly one element and all elements in A are covered. Thus the mapping must be one-to-one

## Functions on real numbers

**Definition**: Let f1 and f2 be functions from A to **R** (reals). Then f1 + f2 and f1 * f2 are also functions from A to **R** defined by

- (f1 + f2)(x) = f1(x) + f2(x)
- (f1 * f2)(x) = f1(x) * f2(x).

**Examples:**

- **Assume**
  - f1(x) = x - 1
  - f2(x) = $x^3$ + 1

  **then**
  - (f1 + f2)(x) = $x^3 + x$
  - (f1 * f2)(x) = $x^4 - x^3 + x - 1$.



## Increasing and decreasing functions

**Definition**: A function f whose domain and codomain are subsets of real numbers is **strictly increasing** if f(x) > f(y) whenever x > y and x and y are in the domain of f. Similarly, f is called **strictly decreasing** if f(x) < f(y) whenever x > y and x and y are in the domain of f.

**Note:** Strictly increasing and strictly decreasing functions are one-to-one.

**Example:**

- Let g : **R** → **R**, where g(x) = 2x - 1. Is it increasing ?

## Increasing and decreasing functions

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**Note:** Strictly increasing and strictly decreasing functions are one-to-one (injective).

**Example:**

- Let $g : \mathbf{R} \to \mathbf{R}$, where $g(x) = 2x - 1$. Is it increasing ?
- **Proof .**

  **For** $x>y$ holds $2x > 2y$ and subsequently $2x-1 > 2y-1$

  **Thus g is strictly increasing.**



## Identity function

**Definition**: Let A be a set. The **identity function** on A is the function $i_A: A \to A$ where $i_A(x) = x$.

**Example:**

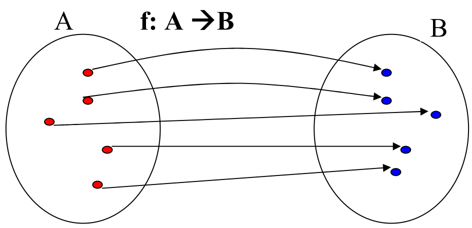
- Let $A = \{1,2,3\}$

**Then:**

- $i_A(1) = 1$
- $i_A(2) = 2$
- $i_A(3) = 3$.

# Bijective functions

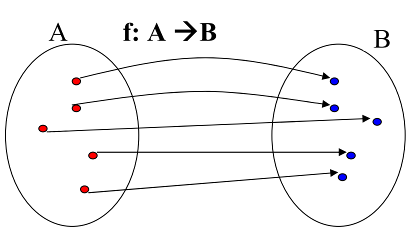
**Definition:** A function f is called **a bijection** if it is **both one-to-one and onto**.






# Inverse functions

**Definition:** Let f be a **bijection** from set A to set B. The **inverse function of f** is the function that assigns to an element b from B the unique element a in A such that f(a) = b. The inverse function of f is denoted by $\mathbf{f^{-1}}$. Hence, $f^{-1}(b) = a$, when f(a) = b. If the inverse function of f exists, f is called **invertible.**




**f is bijective**

# Inverse functions

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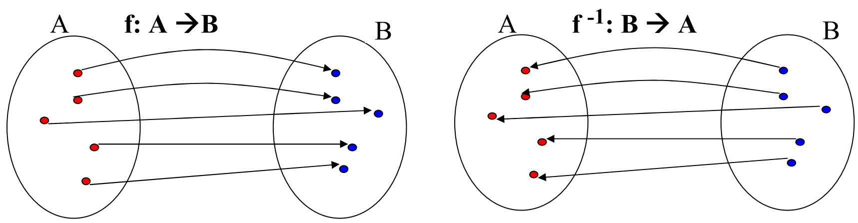



**f is bijective** | **Inverse of f**

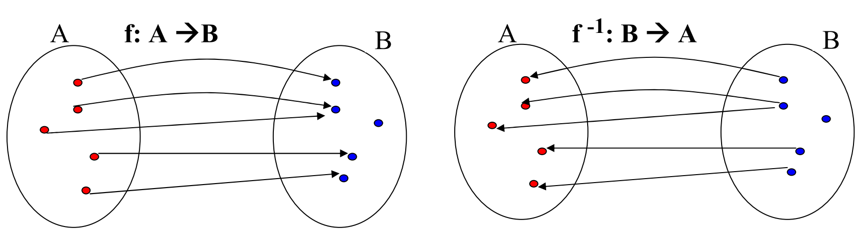


# Inverse functions

Note: if f is not a bijection then it is not possible to define the inverse function of f. **Why?**

**Assume f is not one-to-one:**

?



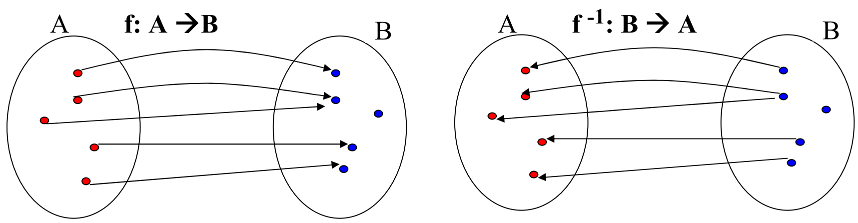

**f** | **'Inverse'**

# Inverse functions

Note: if f is not a bijection then it is not possible to define the inverse function of f. **Why?**

**Assume f is not one-to-one:**

Inverse is not a function. One element of B is mapped to two different elements.




**f** **'Inverse'**

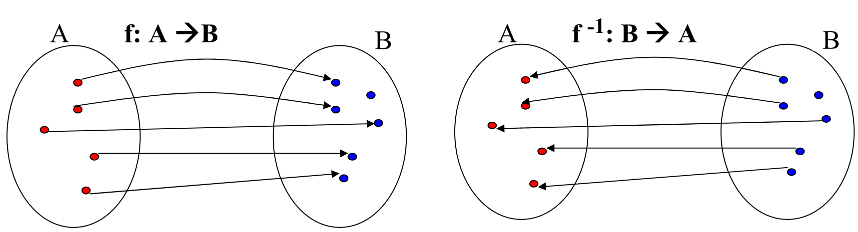


# Inverse functions

Note: if f is not a bijection then it is not possible to define the inverse function of f. Why?

**Assume f is not onto:**

?



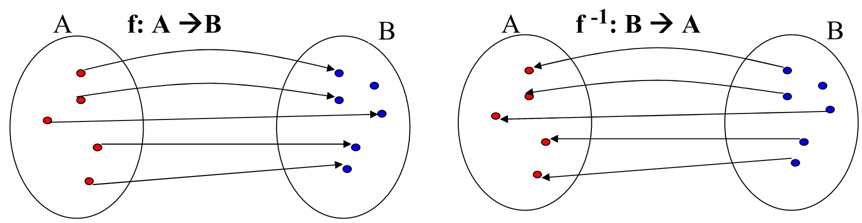

**f** **'Inverse'**

# Inverse functions

Note: if f is not a bijection then it is not possible to define the inverse function of f. Why?

**Assume f is not onto:**

Inverse is not a function. One element of B is not assigned any value in B.




**f** **'Inverse'**



# Inverse functions

**Example 1:**

- Let A = {1,2,3} and $i_A$ be the identity function

- $i_A(1) = 1$ $i_A^{-1}(1) = 1$
- $i_A(2) = 2$ $i_A^{-1}(2) = 2$
- $i_A(3) = 3$ $i_A^{-1}(3) = 3$

- Therefore, the inverse function of $i_A$ is $i_A$.

## Inverse functions

**Example 2:**

- Let g : **R** → **R**, where g(x) = 2x - 1.
- What is the inverse function $g^{-1}$ ?

**Approach to determine the inverse:**

y = 2x - 1 => y + 1 = 2x

=> (y+1)/2 = x

- Define $\mathbf{g^{-1}(y) = x = (y+1)/2}$

**Test the correctness of inverse:**

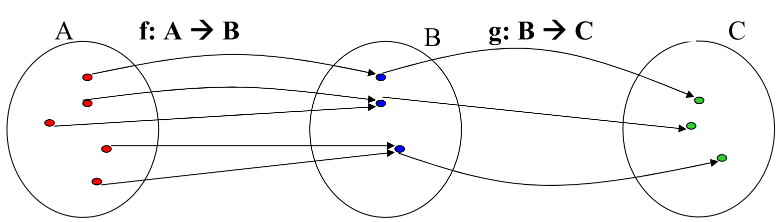
- **g(3)** = 2*3 - 1 = **5**
- $\mathbf{g^{-1}}$ **(5)** = (5+1)/2 = **3**
- g(**10**) = 2*10 - 1 = **19**
- $g^{-1}$ (19) = (19+1)/2 = **10**.



## Composition of functions

**Definition**: Let f be a function from set A to set B and let g be a function from set B to set C. The **composition of the functions g and f**, denoted by g ○ f is defined by

- **(g ○ f)(a) = g(f(a)).**



# Composition of functions

**Example 1:**

- Let A = {1,2,3} and B = {a,b,c,d}

| | |
|---|---|
| $g : A \rightarrow A$, | $f: A \rightarrow B$ |
| $1 \rightarrow 3$ | $1 \rightarrow b$ |
| $2 \rightarrow 1$ | $2 \rightarrow a$ |
| $3 \rightarrow 2$ | $3 \rightarrow d$ |

$f \circ g : A \rightarrow B$:

- $1 \rightarrow d$
- $2 \rightarrow b$
- $3 \rightarrow a$



# Composition of functions

**Example 2:**

- Let f and g be function from Z into Z, where
- $f(x) = 2x$ and $g(x) = x^2$.
- $f \circ g : Z \rightarrow Z$
- $(f \circ g)(x) = f(g(x))$
  $= f( x^2 )$
  $= 2(x^2)$
- $g \circ f : Z \rightarrow Z$
- $(g \circ f)(x) = g(f(x))$
  $= g(2x)$
  $= (2x)^2$
  $= 4x^2$

# Composition of functions

**Example 3:**

- **$(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$, for all x.**
- Let $f : \mathbf{R} \to \mathbf{R}$, where $f(x) = 2x - 1$ and $f^{-1}(x) = (x+1)/2$.
- $(f \circ f^{-1})(x) = f(f^{-1}(x))$
  $= f( (x+1)/2 )$
  $= 2( (x+1)/2 ) - 1$
  $= (x+1) - 1$
  $= x$
- $(f^{-1} \circ f)(x) = f^{-1}(f(x))$
  $= f^{-1}( 2x - 1 )$
  $= (2x)/2$
  $= x$



# Some functions

**Definitions:**

- The **floor function** assigns a real number x the largest integer that is less than or equal to x. The floor function is denoted by $\lfloor x \rfloor$.
- The **ceiling function** assigns to the real number x the smallest integer that is greater than or equal to x. The ceiling function is denoted by $\lceil x \rceil$.

Other important functions:

- Factorials: $n! = n(n-1)$ such that $1! = 1$

# Sequences and summations



## Sequences

**Definition**: A **sequence** is a function from a subset of the set of integers (typically the set {0,1,2,...} or the set {1,2,3,...} to a set S. We use the notation $a_n$ to denote the image of the integer n. We call $a_n$ a term of the sequence.

**Notation:** $\{a_n\}$ is used to represent the sequence (note {} is the same notation used for sets, so be careful). $\{a_n\}$ represents the ordered list $a_1, a_2, a_3, \ldots$ .

1 2 3 4 5 6 ….

↓ ↓ ↓ ↓ ↓ ↓

$a_1$ $a_2$ $a_3$ $a_4$ $a_5$ $a_6$ ….

$\{a_n\}$

## Sequences

**Examples:**

- (1) $a_n = n^2$, where $n = 1,2,3...$
  - What are the elements of the sequence?

    1, 4, 9, 16, 25, ...
- (2) $a_n = (-1)^n$, where n=0,1,2,3,...
  - Elements of the sequence?

    1, -1, 1, -1, 1, ...
- 3) $a_n = 2^n$, where n=0,1,2,3,...
  - Elements of the sequence?

    1, 2, 4, 8, 16, 32, ...



## Arithmetic progression

**Definition:** An **arithmetic progression** is a sequence of the form

a, a+d,a+2d, …, a+nd

where a is the *initial term* and d is *common difference*, such that both belong to R.

**Example:**

- $s_n = -1+4n$ for n=0,1,2,3, …
- members: -1, 3, 7, 11, …

# Geometric progression

**Definition** A **geometric progression** is a sequence of the form:

$a, ar, ar^2, ..., ar^k,$

where a is the *initial term*, and r is the *common ratio*. Both a and r belong to R.

**Example:**

- $a_n = (\frac{1}{2})^n$ for $n = 0,1,2,3, \ldots$
  members: 1,½, ¼, 1/8, …..



# Sequences

- Given a sequence finding a rule for generating the sequence is not always straightforward

**Example:**

- Assume the sequence: 1,3,5,7,9, ….
- What is the formula for the sequence?
- Each term is obtained by adding 2 to the previous term.
- 1, 1+2=3, 3+2=5, 5+2=7
- It suggests **an arithmetic progression**: a+nd
  with a=1 and d=2
    - $a_n$=1+2n or $a_n$=1+2n

# Sequences

- Given a sequence finding a rule for generating the sequence is not always straightforward

**Example 2:**

- Assume the sequence: 1, 1/3, 1/9, 1/27, …
- What is the sequence?
- The denominators are powers of 3.

  1, 1/3= 1/3, (1/3)/3=1/(3*3)=1/9, (1/9)/3=1/27
- This suggests a **geometric progression**: $ar^k$

  with a=1 and r=1/3
    - $(1/3)^n$