# CS 441 Discrete Mathematics for CS
# Lecture 7

# Sets and set operations

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## Course administration

**Thursday lecture:**
- As scheduled

**Recitations:**
- As scheduled

- But please watch for university-wide announcements regarding class cancelations for Thursday and Friday

**Homework 3:**
- We will accept homework submissions by Friday noon
- If not submitting during the class on Thursday, please submit directly to Vyasa Sai, the TA for the course
- If you anticipate a problem of getting in on Thursday or Friday please submit your homework early

# Course administration

**Midterm 1:**

- Week of October 4, 2009
- Covers chapter 1 of the textbook
- Closed book
- Tables for equivalences and rules of inference will be given to you

**Course web page:**

http://www.cs.pitt.edu/~milos/courses/cs441/



# Set

- **Definition**: A **set** is a (unordered) collection of objects. These objects are sometimes called **elements** or **members** of the set. (Cantor's naive definition)

- **Examples:**
  - **Vowels in the English alphabet**

    V = { a, e, i, o, u }
  - **First seven prime numbers**.

    X = { 2, 3, 5, 7, 11, 13, 17 }

## Representing sets

**Representing a set by:**

**1) Listing (enumerating) the members of the set.**

**2) Definition by property, using the set builder notation**

{x| x has property P}.

**Example:**

- Even integers between 50 and 63.

**1) E = {50, 52, 54, 56, 58, 60, 62}**

**2) E = {x| 50 <= x < 63, x is an even integer}**

If enumeration of the members is hard we often use ellipses.

**Example:** a set of integers between 1 and 100

- **A= {1,2,3 …, 100}**



## Important sets in discrete math

- **Natural numbers:**
  - **N** = {0,1,2,3, …}
- **Integers**
  - **Z** = {…, -2,-1,0,1,2, …}
- **Positive integers**
  - $\mathbf{Z}^+$ = {1,2, 3….}
- **Rational numbers**
  - **Q** = $\{p/q \mid p \in Z, q \in Z, q \neq 0\}$
- **Real numbers**
  - **R**

## Russell's paradox

Cantor's naive definition of sets leads to Russell's paradox:

- **Let S = { x | x $\notin$ x },**
  **is a set of sets that are not members of themselves.**
- **Question: Where does the set S belong to?**
  - Is S $\in$ S or S $\notin$ S?
- **Cases**
  - **S $\in$ S ?**



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- **Cases**
  - **S $\in$ S ?:** it does not satisfy the condition so it must hold that **S $\notin$ S**
  - **S $\notin$ S ?:**

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- **Question: Where does the set S belong to?**
  - Is S $\in$ S or S $\notin$ S?
- **Cases**
  - **S $\in$ S ?:** S does not satisfy the condition so it must hold that **S $\notin$ S (or S $\in$ S does not hold)**
  - **S $\notin$ S ?:** S is included in the set S and hence S $\notin$ S does not hold
- **A paradox: we cannot decide if S belongs to S or not**
- **Russell's answer: theory of types – used for sets of sets**



# Equality

**Definition:** Two sets are equal if and only if they have the same elements.

**Example:**

- **{1,2,3} = {3,1,2} = {1,2,1,3,2}**

**Note:** Duplicates don't contribute anything new to a set, so remove them. The order of the elements in a set doesn't contribute anything new.

**Example:** Are {1,2,3,4} and {1,2,2,4} equal?

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**Note:** Duplicates don't contribute anything new to a set, so remove them. The order of the elements in a set doesn't contribute anything new.

**Example:** Are {1,2,3,4} and {1,2,2,4} equal?

**No!**

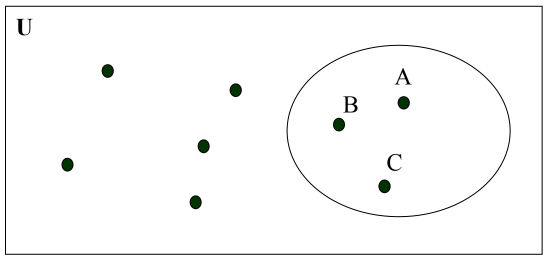


## Special sets

- **Special sets:**
  - **The universal set** is denoted by **U:** the set of all objects under the consideration.
  - **The empty set** is denoted as ∅ **or { }.**

## Venn diagrams

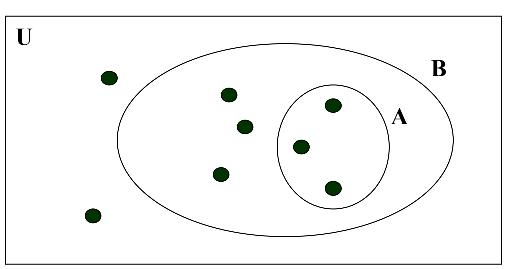
- A set can be visualized using **Venn Diagrams**:
  - V={ A, B, C }






## A Subset

- **Definition**: A **set** A is said to be a **subset** of B **if and only if** every element of A is also an element of B. We use **A ⊆ B** to indicate **A is a subset of B**.




- Alternate way to define A is a subset of B:

  $\forall x \, (x \in A) \rightarrow (x \in B)$

## Empty set/Subset properties

**Theorem** $\emptyset \subseteq$ **S**

- **Empty set is a subset of any set.**

**Proof:**

- Recall the definition of a subset: all elements of a set A must be also elements of B: $\forall x\ (x \in A) \rightarrow (x \in B)$.
- We must show the following implication holds for any S
  $\forall x\ (x \in \emptyset) \rightarrow (x \in S)$
- ?



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**Proof:**

- Recall the definition of a subset: all elements of a set A must be also elements of B: $\forall x\ (x \in A \rightarrow x \in B)$.
- We must show the following implication holds for any S
  $\forall x\ (x \in \emptyset \rightarrow x \in S)$
- Since the empty set does not contain any element, $x \in \emptyset$ is **always False**
- Then the implication is **always True**.

**End of proof**

## Subset properties

**Theorem:** $S \subseteq S$

- **Any set S is a subset of itself**

**Proof:**

- the definition of a subset says: all elements of a set A must be also elements of B: $\forall x\ (x \in A) \rightarrow (x \in B)$.
- Applying this to S we get:
- $\forall x\ (x \in S) \rightarrow (x \in S)$ …



## Subset properties

**Theorem:** $S \subseteq S$

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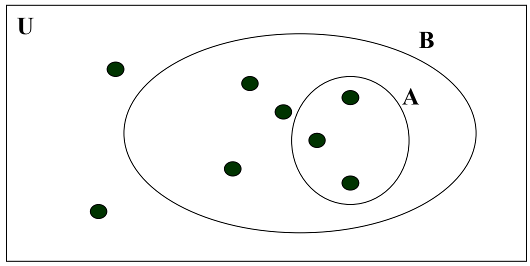
- Applying this to S we get:
- $\forall x\ (x \in S \rightarrow x \in S)$ which is trivially **True**
- End of proof

**Note on equivalence:**

- Two sets are equal if each is a subset of the other set.

# A proper subset

**Definition**: A **set A** is said to be a **proper subset** of B if and only if **A ⊆ B** and **A ≠ B**. We denote that A is a proper subset of B with the notation A ⊂ B.



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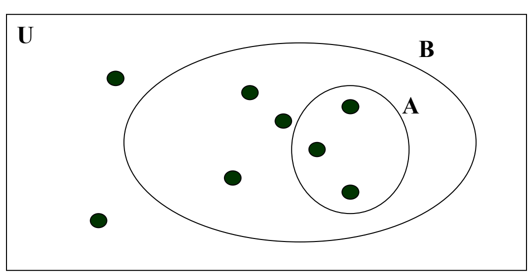
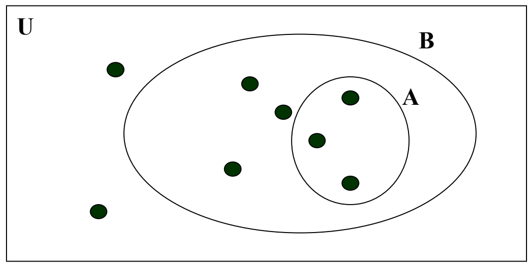



**Example:** A={1,2,3} B ={1,2,3,4,5}

Is: A ⊂ B ?

# A proper subset

**Definition**: A **set A** is said to be a **proper subset** of B if and only if $\mathbf{A \subseteq B}$ and $\mathbf{A \neq B}$. We denote that A is a proper subset of B with the notation $A \subset B$.




**Example:** A={1,2,3} B ={1,2,3,4,5}

Is: $A \subset B$ ? Yes.



# Cardinality

**Definition:** Let S be a set. If there are exactly n distinct elements in S, where n is a nonnegative integer, we say S is a finite set and that n is the **cardinality of S**. The cardinality of S is denoted by **| S |.**

**Examples:**

- **V={1 2 3 4 5}**

  **| V | = ?**

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**Examples:**

- **V={1 2 3 4 5}**
  **| V | = 5**
- **A={1,2,3,4, …, 20}**
  **|A| =?**



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- **A={1,2,3,4, …, 20}**
  **|A| =20**
- **| ∅ | = ?**

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  **| V | = 5**
- **A={1,2,3,4, …, 20}**
  **|A| =20**
- **| ∅ | = 0**



## Infinite set

**Definition**: A set is **infinite** if it is not finite.

**Examples:**

- The set of natural numbers is an infinite set.
- N = {1, 2, 3, ... }
- The set of reals is an infinite set.

## Power set

**Definition:** Given a set S, the **power set** of S is the set of all subsets of S. The power set is denoted by **P(S).**

**Examples:**

- Assume an empty set ∅
- What is the power set of ∅ ? P(∅) = ?



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- Assume an empty set ∅
- What is the power set of ∅ ? P(∅) = { ∅ }
- What is the cardinality of P(∅) ? | P(∅) | = ?

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- Assume an empty set ∅
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- What is the cardinality of P(∅) ? | P(∅) | = 1.

- Assume set {1}
- P( {1} ) = ?



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- |P({1})| = ?

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- P( {1} ) = { ∅, {1} }
- |P({1})| = 2



## Power set

- P( {1} ) = { ∅, {1} }
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- P( {1,2} ) = { ∅, {1}, {2}, {1,2} }
- |P({1,2} )| =?



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- Assume {1,2}
- P( {1,2} ) = { ∅, {1}, {2}, {1,2} }
- |P({1,2} )| =4

- Assume {1,2,3}
- P({1,2,3}) = ?

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- Assume {1,2}
- P( {1,2} ) = { ∅, {1}, {2}, {1,2} }
- |P({1,2} )| =4

- Assume {1,2,3}
- P({1,2,3}) = {∅, {1}, {2}, {3}, {1,2}, {1,3}, {2,3}, {1,2,3} }
- |P({1,2,3} | = ?



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- P( {1,2} ) = { ∅, {1}, {2}, {1,2} }
- |P({1,2} )| =4

- Assume {1,2,3}
- P({1,2,3}) = {∅, {1}, {2}, {3}, {1,2}, {1,3}, {2,3}, {1,2,3} }
- |P({1,2,3} | = 8

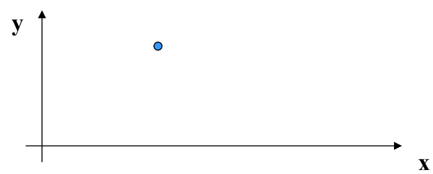
- **If S is a set with |S| = n then | P(S) | = $2^n$..**

## N-tuple

- Sets are used to represent unordered collections.
- Ordered-n tuples are used to represent an ordered collection.

**Definition**: An **ordered n-tuple** (x1, x2, ..., xN) is the ordered collection that has x1 as its first element, x2 as its second element, ..., and xN as its N-th element, $N \geq 2$.

**Example:**




- Coordinates of a point in the 2-D plane (12, 16)



## Cartesian product

**Definition**: Let S and T be sets. The **Cartesian product of S and T**, denoted by **S x T,** is the set of all ordered pairs (s,t), where $s \in S$ and $t \in T$. Hence,

- $S \times T = \{ (s,t) \mid s \in S \wedge t \in T \}$.

**Examples:**

- S = {1,2} and T = {a,b,c}
- S x T = { (1,a), (1,b), (1,c), (2,a), (2,b), (2,c) }
- T x S = { (a,1), (a, 2), (b,1), (b,2), (c,1), (c,2) }
- Note: $S \times T \neq T \times S$ **!!!!**

## Cardinality of the Cartesian product

- |S x T| = |S| * |T|.

**Example:**

- A= {John, Peter, Mike}
- B ={Jane, Ann, Laura}
- A x B=



## Cardinality of the Cartesian product

- |S x T| = |S| * |T|.

**Example:**

- A= {John, Peter, Mike}
- B ={Jane, Ann, Laura}
- A x B= {(John, Jane),(John, Ann) , (John, Laura), (Peter, Jane), (Peter, Ann) , (Peter, Laura) , (Mike, Jane) , (Mike, Ann) , (Mike, Laura)}
- |A x B| =

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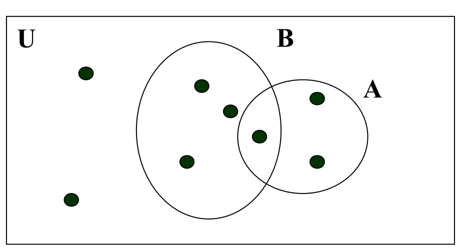
- |A x B| = 9
- |A|=3, |B|=3 → |A| |B|= 9

**Definition:** A subset of the Cartesian product A x B is called a relation from the set A to the set B.



## Set operations

**Definition**: Let A and B be sets. The **union of A and B**, denoted by $A \cup B$, is the set that contains those elements that are either in A or in B, or in both.
- Alternate: $A \cup B = \{ x \mid x \in A \vee x \in B \}$.




- **Example:**
- A = {1,2,3,6}  B = { 2,4,6,9}
- $A \cup B$ = ?

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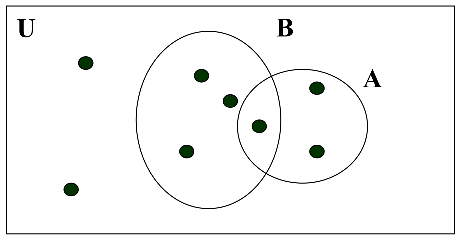


- **Example:**
- $A = \{1,2,3,6\}$ $B = \{ 2,4,6,9\}$
- $A \cup B = \{ 1,2,3,4,6,9 \}$



# Set operations

**Definition**: Let A and B be sets. The **intersection of A and B**, denoted by $A \cap B$, is the set that contains those elements that are in both A and B.

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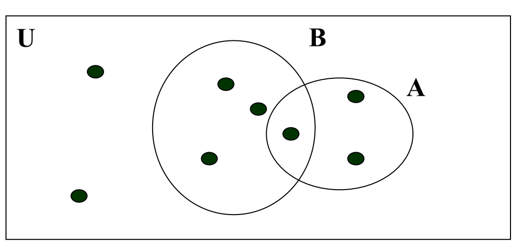


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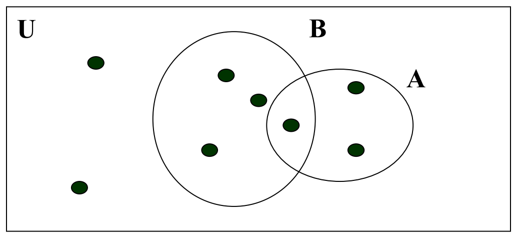



Example:

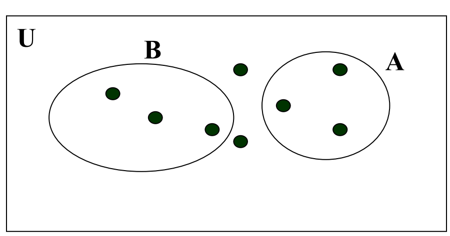
- $A = \{1,2,3,6\}$ $B = \{ 2, 4, 6, 9\}$
- $A \cap B = \{ 2, 6 \}$



# Disjoint sets

**Definition**: Two sets are called **disjoint** if their intersection is empty.

- Alternate: A and B are disjoint **if and only if** $A \cap B = \emptyset$.




**Example:**

- $A=\{1,2,3,6\}$ $B=\{4,7,8\}$ Are these disjoint?

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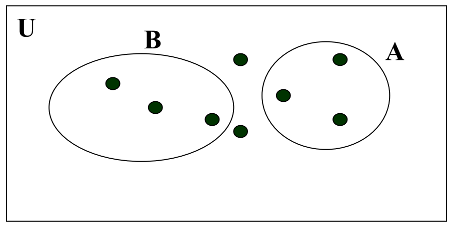



**Example:**

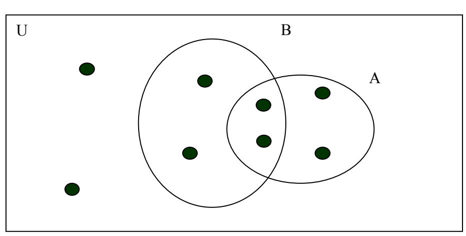
- A={1,2,3,6}  B={4,7,8}  Are these disjoint?
- Yes.
- $A \cap B = \varnothing$



# Cardinality of the set union

**Cardinality of the set union.**

- $|A \cup B| = |A| + |B| - |A \cap B|$




- Why this formula?

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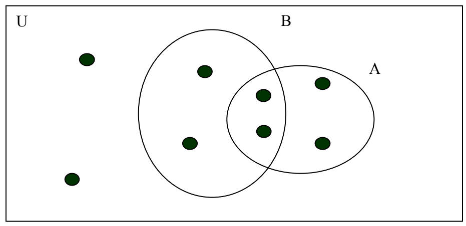


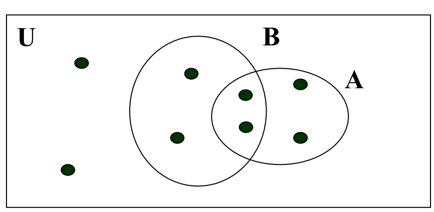

- Why this formula? Correct for an over-count.
- More general rule:
  - **The principle of inclusion and exclusion.**



# Set difference

**Definition**: Let A and B be sets. The **difference of A and B**, denoted by **A - B**, is the set containing those elements that are in A but not in B. The difference of A and B is also called the complement of B with respect to A.

- Alternate: $A - B = \{ x \mid x \in A \wedge x \notin B \}$.




**Example:** A= {1,2,3,5,7} B = {1,5,6,8}

- A - B =?

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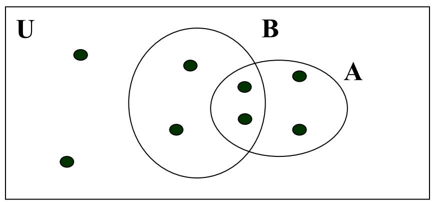



**Example:** A= {1,2,3,5,7} B = {1,5,6,8}

- A - B ={2,3,7}

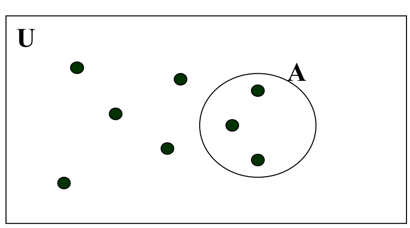


## Complement of a set

**Definition:** Let U be the **universal set**: the set of all objects under the consideration.

**Definition:** The **complement of the set A**, denoted by $\bar{A}$, is the complement of A with respect to U.

- Alternate: $\bar{A} = \{ x \mid x \notin A \}$




**Example:** U={1,2,3,4,5,6,7,8} A ={1,3,5,7}

- $\bar{A}$ = ?

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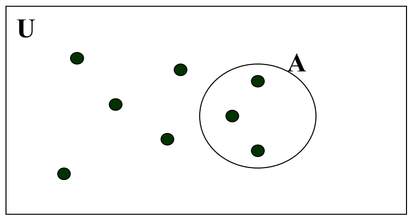


**Example:** U={1,2,3,4,5,6,7,8} A ={1,3,5,7}

- $\bar{A} = \{2,4,6,8\}$



## Set identities

**Set Identities (analogous to logical equivalences)**

- **Identity**
  - $A \cup \emptyset = A$
  - $A \cap U = A$
- **Domination**
  - $A \cup U = U$
  - $A \cap \emptyset = \emptyset$
- **Idempotent**
  - $A \cup A = A$
  - $A \cap A = A$

## Set identities

- **Double complement**
  - $\overline{\overline{A}} = A$
- **Commutative**
  - $A \cup B = B \cup A$
  - $A \cap B = B \cap A$
- **Associative**
  - $A \cup (B \cup C) = (A \cup B) \cup C$
  - $A \cap (B \cap C) = (A \cap B) \cap C$
- **Distributive**
  - $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
  - $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$



## Set identities

- **DeMorgan**
  - $\overline{(A \cap B)} = \overline{A} \cup \overline{B}$
  - $\overline{(A \cup B)} = \overline{A} \cap \overline{B}$
- **Absorbtion Laws**
  - $A \cup (A \cap B) = A$
  - $A \cap (A \cup B) = A$
- **Complement Laws**
  - $A \cup \overline{A} = U$
  - $A \cap \overline{A} = \varnothing$