CS 441 Discrete Mathematics for CS
Lecture 6

Informal proofs. Types of proofs.

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Course administration

- **Homework 2 is due today**
- **Homework 3:**
- **out today and due on September 24, 2009**
- **Recitations tomorrow**
  will cover topics/problems related to Homework 3
- **Course web page:**
  http://www.cs.pitt.edu/~milos/courses/cs441/

Proofs

- The truth value of some statement about the world is obvious and easy to assign
- The truth of other statements may not be obvious, …

  …. But it may still follow (be derived) from known facts about the world

To show the truth value of such a statement follows from other statements we need to provide **a correct supporting argument**

- **a proof**

Theorems and proofs

- **Theorem:** a statement that can be shown to be true.

Typically the theorem looks like this:

$(p1 \wedge p2 \wedge p3 \wedge \ldots \wedge pn) \rightarrow q$

Premises (hypotheses) — **conclusion**

- **Example:**

  Fermat's Little theorem:

  – If p is a prime and a is an integer not divisible by p, then: $a^{p-1} \equiv 1 \bmod p$

Theorems and proofs

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  Premises (hypotheses) **conclusion**

Formal proofs

Proof:

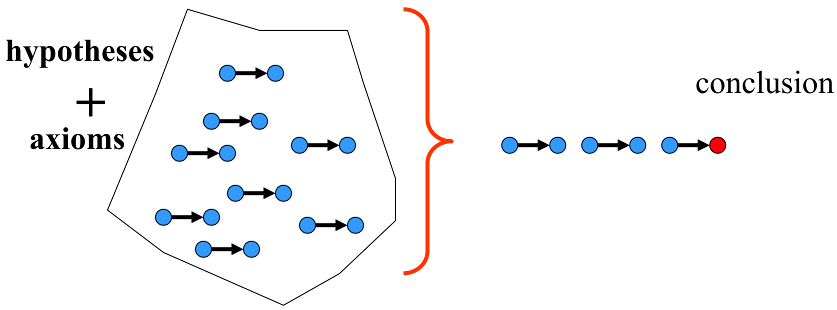
- Provides an argument supporting the validity of the statement
- Proof of the theorem:
  - shows that the conclusion follows from premises
  - may use:
    - Premises
    - Axioms
    - Results of other theorems

Formal proofs:

- steps of the proofs **follow logically** from the set of premises and axioms

Formal proofs

- **Formal proofs:**
  - show that steps of the proofs follow logically from the set of hypotheses and axioms



In the class we assume formal proofs in the propositional logic

Rules of inference

Rules of inference: **logically valid inference patterns**

Example;

- **Modus Ponens**, or the Law of Detachment
- Rule of inference

$$\begin{array}{l} p \\ \underline{p \rightarrow q} \\ \therefore\ q \end{array}$$

- Given p is true and the implication $p \rightarrow q$ is true then q is true.

p	q	$p \rightarrow q$
False	*False*	*True*
False	*True*	*True*
True	*False*	*False*
True	*True*	*True*

Proofs using rules of inference

Translations:

- **Assumptions:** $\neg p \wedge q$, $r \rightarrow p$, $\neg r \rightarrow s$, $s \rightarrow t$
- **Hypothesis:** t

Proof:

- 1. $\neg p \wedge q$ Hypothesis
- 2. $\neg p$ Simplification
- 3. $r \rightarrow p$ Hypothesis
- 4. $\neg r$ Modus tollens (step 2 and 3)
- 5. $\neg r \rightarrow s$ Hypothesis
- 6. s Modus ponens (steps 4 and 5)
- 7. $s \rightarrow t$ Hypothesis
- 8. t Modus ponens (steps 6 and 7)
- **end of proof**

Informal proofs

Proving theorems in practice:

- The steps of the proofs are not expressed in any formal language as e.g. propositional logic
- Steps are argued less formally using English, mathematical formulas and so on
- One step in the proof may consist of multiple derivations, portions of the proof may be skipped or assumed correct,
- axioms may not be explicitly stated.
- One must always watch the consistency of the argument made, logic and its rules can often help us to decide the soundness of the argument if it is in question

We use (informal) proofs to illustrate different methods of proving theorems

Methods of proving theorems

General methods to prove the theorems:

- **Direct proof**
  - $p \rightarrow q$ is proved by showing that if p is true then q follows
- **Indirect proof**
  - Show the contrapositive $\neg q \rightarrow \neg p$. If $\neg q$ holds then $\neg p$ follows
- **Proof by contradiction**
  - Show that $(p \wedge \neg q)$ contradicts the assumptions
- **Proof by cases**
- **Proofs of equivalence**
  - $p \leftrightarrow q$ is replaced with $(p \rightarrow q) \wedge (q \rightarrow p)$

Sometimes one method of proof does not go through as nicely as the other method. You may need to try more than one approach.

Direct proof

- $p \rightarrow q$ is proved by showing that if p is true then q follows
- **Example:** Prove that "If n is odd, then n^2 is odd."

Proof:

- Assume the premise (hypothesis) is true, i.e. suppose n is odd.
- Then $n = 2k + 1$, where k is an integer.

Direct proof

- $p \rightarrow q$ is proved by showing that if p is true then q follows
- **Example:** Prove that "If n is odd, then n^2 is odd."

Proof:

- Assume the hypothesis is true, i.e. suppose n is odd.
- Then $n = 2k + 1$, where k is an integer.

$$n^2 = (2k+1)^2$$
$$= 4k^2 + 4k + 1$$
$$= 2(2k^2 + 2k) + 1$$

Direct proof

- $p \rightarrow q$ is proved by showing that if p is true then q follows
- **Example:** Prove that "If n is odd, then n^2 is odd."

Proof:

- Assume the hypothesis is true, i.e. suppose n is odd.
- Then $n = 2k + 1$, where k is an integer.

$$n^2 = (2k+1)^2$$
$$= 4k^2 + 4k + 1$$
$$= 2(2k^2 + 2k) + 1$$

- Therefore, n^2 is odd. □

Direct proof

- Direct proof may not be the best option. It may become hard to prove the conclusion follows from the premises.

Example: Prove If $3n + 2$ is odd then **n is odd**.

Proof:

- Assume that $3n + 2$ is odd,
  - thus $3n + 2 = 2k + 1$ for some k.
- Then $n = (2k – 1)/3$
- ?

Direct proof

- Direct proof may not be the best option. It may become hard to prove the conclusion follows from the premises.

Example: Prove If $3n + 2$ is odd then **n is odd**.

Proof:

- Assume that $3n + 2$ is odd,
  - thus $3n + 2 = 2k + 1$ for some k.
- Then $n = (2k – 1)/3$
- **Not clear how to continue**

Indirect proof

- To show $p \rightarrow q$ prove its contrapositive $\neg q \rightarrow \neg p$
- Why is this correct?

Indirect proof

- To show $p \rightarrow q$ prove its contrapositive $\neg q \rightarrow \neg p$
- Why? **$p \rightarrow q$ and $\neg q \rightarrow \neg p$ are equivalent !!!**
- Assume $\neg q$ is true, show that $\neg p$ is true.

Example: Prove If $3n + 2$ is odd then n is odd.

Proof:

Contrapositive: If n is even then $3n + 2$ is even.

Proof:

Indirect proof

- To show $p \rightarrow q$ prove its contrapositive $\neg q \rightarrow \neg p$
- Why? **$p \rightarrow q$ and $\neg q \rightarrow \neg p$ are equivalent !!!**
- Assume $\neg q$ is true, show that $\neg p$ is true.

Example: Prove If $3n + 2$ is odd then **n is odd**.

Proof:

- **Contrapositive:** If n is even then $3n + 2$ is even.
- Assume **n is even**, that is $n = 2k$, where k is an integer.

Indirect proof

- To show $p \rightarrow q$ prove its contrapositive $\neg q \rightarrow \neg p$
- Why? **$p \rightarrow q$ and $\neg q \rightarrow \neg p$ are equivalent !!!**
- Assume $\neg q$ is true, show that $\neg p$ is true.

Example: Prove If $3n + 2$ is odd then n is odd.

Proof:

- **Contrapositive:** If n is even then $3n + 2$ is even.
- Assume n is even, that is $n = 2k$, where k is an integer.
- Then: $$\begin{aligned} 3n + 2 &= 3(2k) + 2 \\ &= 6k + 2 \\ &= 2(3k+1) \end{aligned}$$

Indirect proof

- To show $p \rightarrow q$ prove its contrapositive $\neg q \rightarrow \neg p$
- Why? **$p \rightarrow q$ and $\neg q \rightarrow \neg p$ are equivalent !!!**
- Assume $\neg q$ is true, show that $\neg p$ is true.

Example: Prove If **3n + 2 is odd** then n is odd.

Proof:

- **Contrapositive:** If n is even then 3n + 2 is even.
- Assume n is even, that is $n = 2k$, where k is an integer.
- Then: $3n + 2 = 3(2k) + 2$
  $= 6k + 2$
  $= 2(3k+1)$
- Therefore **3n + 2 is even.**

Indirect proof

- To show $p \rightarrow q$ prove its contrapositive $\neg q \rightarrow \neg p$
- Why? **$p \rightarrow q$ and $\neg q \rightarrow \neg p$ are equivalent !!!**
- Assume $\neg q$ is true, show that $\neg p$ is true.

Example: Prove If 3n + 2 is odd then n is odd.

Proof:

- **Contrapositive:** If n is even then 3n + 2 is even.
- Assume n is even, that is $n = 2k$, where k is an integer.
- Then: $3n + 2 = 3(2k) + 2$
  $= 6k + 2$
  $= 2(3k+1)$
- Therefore 3n + 2 is even.
- We proved **¬ "n is odd" → ¬ "3n + 2 is odd".** This is equivalent to **"3n + 2 is odd" → "n is odd".** □

Proof by contradiction

- We want to prove $p \to q$
- The only way to reject (or disprove) $p \to q$ is to show that $(p \wedge \neg q)$ can be true
- However, if we manage to prove that either q or $\neg$ p is True then we contradict $(p \wedge \neg q)$
  - **and subsequently** $p \to q$ must be true
- Proof by contradiction. Show that the assumption $(p \wedge \neg q)$ **leads either to** q or $\neg$ p which generates a contradiction.

Proof by contradiction

- We want to prove $p \to q$
- To reject $p \to q$ show that $(p \wedge \neg q)$ can be true
- To reject $(p \wedge \neg q)$ show that either **q or $\neg$ p** is True

Example: Prove If **3n + 2 is odd** then **n is odd**.

Proof:

- Assume **3n + 2 is odd** and **n is even**, that is n = 2k, where k is an integer.

Proof by contradiction

- We want to prove $p \rightarrow q$
- To reject **$p \rightarrow q$** show that **$(p \wedge \neg q)$** can be true
- To reject **$(p \wedge \neg q)$** show that either **q or $\neg$ p** is True

Example: Prove If **3n + 2 is odd** then **n is odd.**

Proof:

- Assume **3n + 2 is odd** and **n is even**, that is n = 2k, where k is an integer.
- Then: $3n + 2 = 3(2k) + 2$
  $= 6k + 2$
  $= 2(3k + 1)$
- Thus **3n + 2 is…**

Proof by contradiction

- We want to prove $p \rightarrow q$
- To reject **$p \rightarrow q$** show that **$(p \wedge \neg q)$** can be true
- To reject **$(p \wedge \neg q)$** show that either **q or $\neg$ p** is True

Example: Prove If **3n + 2 is odd** then n is odd.

Proof:

- Assume **3n + 2 is odd** and n is even, that is n = 2k, where k an integer.
- Then: $3n + 2 = 3(2k) + 2$
  $= 6k + 2$
  $= 2(3k + 1)$
- Thus **3n + 2 is even.** This is a contradiction with the assumption that **3n + 2 is odd.** Therefore **n is odd**. □

Vacuous proof

We want to show $p \rightarrow q$

- Suppose p (the hypothesis) is always false
- Then $p \rightarrow q$ is always true.

Reason:

- $F \rightarrow q$ is always T, whether q is True or False

Example:

- Let P(n) denotes "if $n > 1$ then $n^2 > n$" is TRUE.
- Show that P(0).

Proof:

- For n=0 the premise is False. Thus P(0) is always true.

Trivial proofs

We want to show $p \rightarrow q$

- Suppose the conclusion q is always true
- Then the implication $p \rightarrow q$ is trivially true.
- **Reason:**
- $p \rightarrow T$ is always T, whether p is True or False

Example:

- Let P(n) is "if a >=b then $a^n >= b^n$"
- Show that P(0)

Proof:

$a^0 >= b^0$ is 1=1 trivially true.

Proof by cases

- We want to show $p1 \vee p2 \vee \ldots \vee pn \rightarrow q$
- Note that this is equivalent to

  $(p1 \rightarrow q) \wedge (p2 \rightarrow q) \wedge \ldots \wedge (pn \rightarrow q)$
- **Why?**

Proof by cases

- We want to show $p1 \vee p2 \vee \ldots \vee pn \rightarrow q$
- Note that this is equivalent to
  - $(p1 \rightarrow q) \wedge (p2 \rightarrow q) \wedge \ldots \wedge (pn \rightarrow q)$
- **Why?**
- $p1 \vee p2 \vee \ldots \vee pn \rightarrow q$ <=> (useful)
- $\neg (p1 \vee p2 \vee \ldots \vee pn) \vee q$ <=> (De Morgan)
- $(\neg p1 \wedge \neg p2 \wedge \ldots \wedge \neg pn) \vee q$ <=> (distributive)
- $(\neg p1 \vee q) \wedge (\neg p2 \vee q) \wedge \ldots \wedge (\neg pn \vee q)$ <=> (useful)
- $(p1 \rightarrow q) \wedge (p2 \rightarrow q) \wedge \ldots \wedge (pn \rightarrow q)$

Proof by cases

We want to show $p1 \vee p2 \vee \ldots \vee pn \rightarrow q$

- Equivalent to $(p1 \rightarrow q) \wedge (p2 \rightarrow q) \wedge \ldots \wedge (pn \rightarrow q)$

Prove individual cases as before. All of them must be true.

Example: Show that |x||y|=|xy|.

Proof:

- 4 cases:
- x >=0, y>=0 xy >0 and |xy|=xy=|x||y|
- x>= 0, y <0 xy < 0 and |xy|=-xy =x (-y)=|x||y|
- x<0, y>=0 xy < 0 and |xy|=-xy =(-x) y=|x||y|
- x<0, , y <0 xy >0 and |xy|= (-x)(-y) =|x||y|
- All cases proved.

Proof of equivalences

We want to prove $p \leftrightarrow q$

- Statements: p if and only if q.
- Note that $p \leftrightarrow q$ is equivalent to $[(p \rightarrow q) \wedge (q \rightarrow p)]$
- Both implications must hold.

Example:

- Integer is odd if and only if n^2 is odd.

Proof of $(p \rightarrow q)$:

- $(p \rightarrow q)$ If n is odd then n^2 is odd
- we use a direct proof
- Suppose n is odd. Then n = 2k + 1,where k is an integer.
- n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1
- Therefore, n^2 is odd.

Proof of equivalences

We want to prove $p \leftrightarrow q$

- Note that $p \leftrightarrow q$ is equivalent to $[(p \rightarrow q) \wedge (q \rightarrow p)]$
- Both implications must hold.

- Integer is odd if and only if n^2 is odd.

Proof of $(q \rightarrow p)$**:**

- $(q \rightarrow p)$: if n^2 is odd then n is odd
- we use an indirect proof $(\neg p \rightarrow \neg q)$ is a contrapositive
- n is even that is n = 2k,
- then n^2 = 4k^2= 2(2k^2)
- Therefore n^2 is even. Done proving the contrapositive.

Since both $(p \rightarrow q)$ and $(q \rightarrow p)$ are true the equivalence is true

Proofs with quantifiers

- **Existence proof**
  - **Constructive**
    - Find an example that shows the statement holds.
  - **Nonconstructive**
    - Show it holds for one example but we do not have the witness example (typically ends with one example or other example)

- **Counterexamples:**
  - Are used to disprove universal statements