CS 441 Discrete Mathematics for CS
Lecture 5

Predicate logic.
Formal proofs.

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Quantified statements in Predicate Logic

- When $\forall$**x P(x)** and $\exists$**x P(x)** are true and false?

Statement	When true?	When false?
$\forall$x P(x)	P(x) true for all x	There is an x where P(x) is false.
$\exists$x P(x)	There is some x for which P(x) is true.	P(x) is false for all x.

Suppose the elements in the universe of discourse can be enumerated as x1, x2, ..., xN then:

- **$\forall$x P(x) is true whenever P(x1) $\wedge$ P(x2) $\wedge$ ... $\wedge$ P(xN) is true**
- **$\exists$x P(x) is true whenever P(x1) $\vee$ P(x2) $\vee$ ... $\vee$ P(xN) is true**.

Order of quantifiers

The order of nested quantifiers matters if quantifiers are of different type

- **$\forall x \exists y\ L(x,y)$ is not the same as $\exists y \forall x\ L(x,y)$**

Example:

- Assume L(x,y) denotes "x loves y"
- Then: $\forall x \exists y\ L(x,y)$
- Translates to: Everybody loves somebody.
- And: $\exists y\ \forall x\ L(x,y)$
- Translates to: There is someone who is loved by everyone.

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Different meaning

- And: $\exists y\ \forall x\ L(x,y)$
- Translates to: There is someone who is loved by everyone.

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Order of quantifiers

The order of nested quantifiers does not matter if quantifiers are of the same type

Example:

- For all x and y, if x is a parent of y then y is a child of x
- **Assume:**
  - Parent(x,y) denotes "x is a parent of y"
  - Child(x,y) denotes "x is a child of y"
- Two equivalent ways to represent the statement:
  - $\forall x\ \forall y\ Parent(x,y) \rightarrow Child(y,x)$
  - $\forall y\ \forall x\ Parent(x,y) \rightarrow Child(y,x)$

Translation exercise

Suppose:

- Variables x,y denote people
- L(x,y) denotes "x loves y".

Translate:

- Everybody loves Raymond. $\forall x\ L(x,Raymond)$
- Everybody loves somebody. $\forall x \exists y\ L(x,y)$
- There is somebody whom everybody loves. $\exists y \forall x\ L(x,y)$
- There is somebody who Raymond doesn't love.
  $\exists y \neg L(Raymond,y)$
- There is somebody whom no one loves.
  $\exists y\ \forall x\ \neg L(x,y)$

Negation of quantifiers

English statement:

- Nothing is perfect.
- **Translation:** $\neg \exists x$ Perfect(x)

Another way to express the same meaning:

- **Everything is imperfect.**
- **Translation:** $\forall x \neg$ Perfect(x)

Conclusion: $\neg \exists x\, P(x)$ is equivalent to $\forall x \neg P(x)$

Negation of quantifiers

English statement:

- It is not the case that all dogs are fleabags.
- **Translation:** $\neg \forall x$ Dog(x) $\rightarrow$ Fleabag(x)

Another way to express the same meaning:

- There is a dog that is not a fleabag.
- **Translation:** $\exists x$ Dog(x) $\wedge \neg$ Fleabag(x)

- Logically equivalent to:
  - $\exists x \neg$ (Dog(x) $\rightarrow$ Fleabag(x))

Conclusion: $\neg \forall x\, P(x)$ is equivalent to $\exists x \neg P(x)$

Negation of quantified statements

Negation	Equivalent
$\neg\exists x\ P(x)$	$\forall x\ \neg P(x)$
$\neg\forall x\ P(x)$	$\exists x\ \neg P(x)$

Theorems and proofs

- The truth value of some statement about the world is obvious and easy to assign
- The truth of other statements may not be obvious, …

  …. But it may still follow (be derived) from known facts about the world

To show the truth value of such a statement following from other statements we need to provide **a correct supporting argument**

- **a proof**

Problem:

- It is easy to make a mistake and argue the support incorrectly.

Important questions:

- When is the argument correct?
- How to construct a correct argument, what method to use?

Theorems and proofs

- **Theorem:** a statement that can be shown to be true.
  - **Typically the theorem looks like this:**

    $(p1 \wedge p2 \wedge p3 \wedge \ldots \wedge pn) \rightarrow q$

    Premises (hypotheses) **conclusion**

- **Example:**

  Fermat's Little theorem:
  - If p is a prime and a is an integer not divisible by p, then: $a^{p-1} \equiv 1 \bmod p$

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  Fermat's Little theorem:

  Premises (hypotheses)
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  conclusion

Formal proofs

Proof:

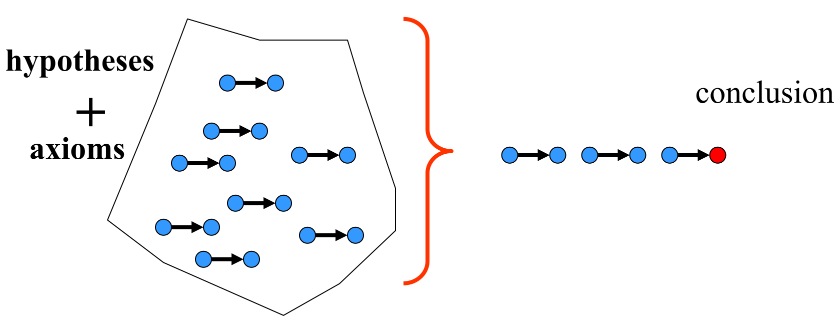
- Provides an argument supporting the validity of the statement
- Proof of the theorem:
  - shows that the conclusion follows from premises
  - may use:
    - Premises
    - Axioms
    - Results of other theorems

Formal proofs:

- steps of the proofs **follow logically** from the set of premises and axioms

Formal proofs

- **Formal proofs:**
  - show that steps of the proofs follow logically from the set of hypotheses and axioms



In the class we assume formal proofs in the propositional logic

Rules of inference

Rules of inference: logically valid inference patterns

Example;

- **Modus Ponens**, or the Law of Detachment
- Rule of inference

$$\begin{array}{l} p \\ \underline{p \rightarrow q} \\ \therefore\ q \end{array}$$

- Given p is true and the implication $p \rightarrow q$ is true then q is true.

Rules of inference

Rules of inference: logically valid inference patterns

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p	q	$p \rightarrow q$
False	*False*	*True*
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Rules of inference

Rules of inference: logically valid inference patterns

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- **Modus Ponens**, or the Law of Detachment
- Rules of inference

$$\begin{array}{l} p \\ \underline{p \rightarrow q} \\ \therefore q \end{array}$$

- Given p is true and the implication $p \rightarrow q$ is true then q is true.
- **Tautology Form:** $(p \wedge (p \rightarrow q)) \rightarrow q$

Rules of inference

- **Addition**

$\mathbf{p \rightarrow (p \vee q)}$

$$\begin{array}{l} \underline{p\qquad\quad} \\ \therefore p \vee q \end{array}$$

- **Example:** It is below freezing now. Therefore, it is below freezing or raining snow.

- **Simplification**

$\mathbf{(p \wedge q) \rightarrow p}$

$$\begin{array}{l} \underline{\ p \wedge q} \\ \therefore p \end{array}$$

- **Example:** It is below freezing and snowing. Therefore it is below freezing.

Rules of inference

- **Modus Tollens**

  $[\neg q \wedge (p \rightarrow q)] \rightarrow \neg p$

  $$\begin{array}{l} \neg q \\ \underline{p \rightarrow q} \\ \therefore \neg p \end{array}$$

- **Hypothetical Syllogism**

  $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$

  $$\begin{array}{l} p \rightarrow q \\ \underline{q \rightarrow r} \\ \therefore p \rightarrow r \end{array}$$

- **Disjunctive Syllogism**

  $[(p \vee q) \wedge \neg p] \rightarrow q$

  $$\begin{array}{l} p \vee q \\ \underline{\neg p} \\ \therefore q \end{array}$$

Rules of inference

- **Logical equivalences (discussed earlier)**

  $A \Leftrightarrow B$

  $A \rightarrow B$ **is a tautology**

Example: De Morgan Law

$\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$

$\neg(p \vee q) \rightarrow \neg p \wedge \neg q$ **is a tautology**

Rules of inference

- A **valid argument** is one built using the rules of inference from premises (hypotheses). When all premises are true the argument leads to a correct conclusion.
- $(p1 \wedge p2 \wedge p3 \wedge \ldots \wedge pn) \rightarrow q$
- **However, if one or more of the premises is false the conclusion may be incorrect.**
- **How to use the rules of inference?**

Applying rules of inference

Assume the following statements (hypotheses):

- It is not sunny this afternoon and it is colder than yesterday.
- We will go swimming only if it is sunny.
- If we do not go swimming then we will take a canoe trip.
- If we take a canoe trip, then we will be home by sunset.

Show that all these lead to a conclusion:

- We will be home by sunset.

Applying rules of inference

Text:

(1) It is not sunny this afternoon and it is colder than yesterday.

(2) We will go swimming only if it is sunny.

(3) If we do not go swimming then we will take a canoe trip.

(4) If we take a canoe trip, then we will be home by sunset.

Propositions:

- p = It is sunny this afternoon, q = it is colder than yesterday, r = We will go swimming , s= we will take a canoe trip
- t= We will be home by sunset

Translation:

- **Assumptions:** (1) $\neg p \wedge q$, (2) $r \rightarrow p$, (3) $\neg r \rightarrow s$, (4) $s \rightarrow t$
- **Hypothesis:** t

Applying rules of inference

- **Approach:**
- p = It is sunny this afternoon, q = it is colder than yesterday, r = We will go swimming , s= we will take a canoe trip
- t= We will be home by sunset
- **Translations:**
- **Assumptions:** $\neg p \wedge q$, $r \rightarrow p$, $\neg r \rightarrow s$, $s \rightarrow t$
- **Hypothesis:** t

Translation: "We will go swimming only if it is sunny".

- Ambiguity: $r \rightarrow p$ or $p \rightarrow r$?
- Sunny is a must before we go swimming
- Thus, if we indeed go swimming it must be sunny, therefore $r \rightarrow p$

Proofs using rules of inference

Translations:

- **Assumptions:** $\neg p \wedge q$, $r \rightarrow p$, $\neg r \rightarrow s$, $s \rightarrow t$
- **Hypothesis:** t

Proof:

- 1. $\neg p \wedge q$ Hypothesis
- 2. $\neg p$ Simplification
- 3. $r \rightarrow p$ Hypothesis
- 4. $\neg r$ Modus tollens (step 2 and 3)
- 5. $\neg r \rightarrow s$ Hypothesis
- 6. s Modus ponens (steps 4 and 5)
- 7. $s \rightarrow t$ Hypothesis
- 8. t Modus ponens (steps 6 and 7)
- **end of proof**