CS 441 Discrete Mathematics for CS
Lecture 4

Predicate logic

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Course administration

- **Homework 1 is due today**
- **Homework 2:**
- **out today and due on September 17, 2009**
- **Recitations tomorrow**
  will cover topics/problems related to Homework 2
- **Course web page:**
  http://www.cs.pitt.edu/~milos/courses/cs441/

Limitations of the propositional logic

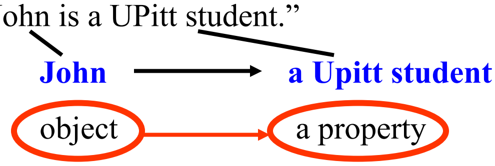
Propositional logic: the world is described in terms of propositions

A proposition is a statement that is either true or false.

Limitations:

(1) **Objects** in elementary statements, **their properties and relations are not explicitly represented** in the propositional logic

- **Example:**
  - "John is a UPitt student."



  - Objects and properties are hidden in the statement, it is not possible to reason about them

Limitations of the propositional logic

(2) Statements for groups of objects require enumeration

- **Example:**
  - If John is a CS UPitt graduate then John has passed cs441

  Translation:

  - John is a CS UPitt graduate → John has passed cs441

  Similar statements can be written for other Upitt graduates:

  - Ann is a CS Upitt graduate → Ann has passed cs441
  - Ken is a CS Upitt graduate → Ken has passed cs441
  - …
- **Solution:** make statements with **variables**
  - If x is a CS Upitt graduate then x has passed cs441
  - x is a CS UPitt graduate → x has passed cs441

Predicate logic

Remedies the limitations of the propositional logic

- Explicitly models objects and their properties
- Allows to make statements with variables and quantify them

Basic building blocks of the predicate logic:

- **Constant** –models a specific object
  Examples: "John", "France", "7"
- **Variable** – represents object of specific type **(defined by the universe of discourse)**
  Examples: x, y
  (universe of discourse can be people, students, numbers)
- **Predicate -** over one, two or many variables or constants.
  – Represents properties or relations among objects
  Examples: Red(car23), student(x), married(John,Ann)

Predicates

Predicates represent properties or relations among objects

A predicate P(x) assigns a value **true or false** to each x depending on whether the property holds or not for x.

- The assignment is best viewed as a big table with the variable x substituted for objects from the universe of discourse

Example:

- Assume **Student(x)** where the universe of discourse are people
- Student(John) …. T (if John is a student)
- Student(Ann) …. T (if Ann is a student)
- Student(Jane) ….. F (if Jane is not a student)
- …

Compound statements in predicate logic

Compound statements are obtained via logical connectives

Examples:

Student(Ann) $\wedge$ Student(Jane)

- **Translation:** "Both Ann and Jane are students"
- **Proposition:** yes.

Country(Sienna) $\vee$ River(Sienna)

- **Translation:** "Sienna is a country or a river"
- **Proposition:** yes.

CS-major(x) $\rightarrow$ Student(x)

- **Translation:** "if x is a CS-major then x is a student"
- **Proposition:** no.

Predicates

Important:

- predicate P(x) is **not a proposition** since there are more objects it can be applied to

This is the same as in propositional logic …

… But the difference is:

- predicate logic allows us to explicitly manipulate and substitute for the objects
- Predicate logic permits quantified sentences where variables are substituted for statements about the group of objects

Quantified statements

Predicate logic lets us to make statements about groups of objects

- To do this we use special quantified expressions

Two types of quantified statements:

- **universal**

  Example: ‘ all CS Upitt graduates have to pass cs441”

  – the statement is true for all graduates
- **existential**

  Example: ‘Some CS Upitt students graduate with honor.’

  – the statement is true for some people

Universal quantifier

Defn: The universal quantification of P(x) is the proposition: "*P(x) is true for all values of x in the domain of discourse*." The notation **∀x P(x)** denotes the universal quantification of P(x), and is expressed as **for every x, P(x).**

Example:

- Let P(x) denote x > x - 1.
- What is the truth value of ∀x P(x)?
- Assume the universe of discourse of x is all real numbers.
- **Answer:** Since every number x is greater than itself minus 1. Therefore, **∀x P(x) is true.**

Universal quantifier

Quantification converts a propositional function into **a proposition** by binding a variable to a set of values from the universe of discourse.

Example:

- Let P(x) denote x > x - 1.
- Is P(x) a proposition? **No.** Many possible substitutions.
- Is ∀x P(x) a proposition? **Yes.** True if for all x from the universe of discourse P(x) is true.

Universally quantified statements

Predicate logic lets us make statements about groups of objects

Universally quantified statement

- CS-major(x) → Student(x)
  - **Translation:** "if x is a CS-major then x is a student"
  - **Proposition:** **no.**
- ∀x CS-major(x) → Student(x)
  - **Translation:** "(For all people it holds that) if a person is a CS-major then she is a student."
  - **Proposition:** **yes.**

Existential quantifier

Definition: The **existential quantification** of P(x) is the proposition "*There exists an element in the domain (universe) of discourse such that P(x) is true*." The notation **∃x P(x)** denotes the existential quantification of P(x), and is expressed as **there is an x such that P(x) is true**.

Example 2:

- Let Q(x) denote x = x + 2 where x is real numbers
- What is the truth value of ∃x Q(x)?
- **Answer:** Since no real number is 2 larger than itself, the truth value of **∃x Q(x) is false**.

Quantified statements

Statements about groups of objects

Example:

- CS-Upitt-graduate (x) ∧ Honor-student(x)
  - **Translation:** "x is a CS-Upitt-graduate and x is an honor student"
  - **Proposition:** **no.**
- ∃ x CS-Upitt-graduate (x) ∧ Honor-student(x)
  - **Translation:** "There is a person who is a CS-Upitt-graduate and who is also an honor student."
  - **Proposition:** ?

Quantified statements

Statements about groups of objects

Example:

- CS-Upitt-graduate (x) ∧ Honor-student(x)
  - **Translation:** "x is a CS-Upitt-graduate and x is an honor student"
  - **Proposition:** **no.**
- ∃ x CS-Upitt-graduate (x) ∧ Honor-student(x)
  - **Translation:** "There is a person who is a CS-Upitt-graduate and who is also an honor student."
  - **Proposition:** **yes**.

Summary of quantified statements

- When ∀**x P(x)** and ∃**x P(x)** are true and false?

Statement	When true?	When false?
∀x P(x)	P(x) true for all x	There is an x where P(x) is false.
∃x P(x)	There is some x for which P(x) is true.	P(x) is false for all x.

Suppose the elements in the universe of discourse can be enumerated as x1, x2, ..., xN then:

- **∀x P(x) is true whenever P(x1) ∧ P(x2) ∧ ... ∧ P(xN) is true**
- **∃x P(x) is true whenever P(x1) ∨ P(x2) ∨ ... ∨ P(xN) is true**.

Translation with quantifiers

Sentence:

- All Upitt students are smart.

- **Assume:** the domain of discourse of x are Upitt students
- **Translation:**
- $\forall x$ Smart(x)

- **Assume:** the universe of discourse are students (all students):
- $\forall x$ at(x,Upitt) $\rightarrow$ Smart(x)

- **Assume:** the universe of discourse are people:
- $\forall x$ student(x) $\wedge$ at(x,Upitt) $\rightarrow$ Smart(x)

Translation with quantifiers

Sentence:

- Someone at CMU is smart.

- **Assume:** the domain of discourse are all CMU affiliates
- **Translation:**
- $\exists$ x Smart(x)

- **Assume:** the universe of discourse are people:
- $\exists$ x at(x,CMU) $\wedge$ Smart(x)

Translation with quantifiers

- **Assume two predicates S(x) and P(x)**

Universal statements typically tie with implications

- **All S(x) is P(x)**
  - $\forall x\ (S(x) \rightarrow P(x))$
- **No S(x) is P(x)**
  - $\forall x(S(x) \rightarrow \neg P(x))$

Existential statements typically tie with conjunctions

- **Some S(x) is P(x)**
  - $\exists x\ (S(x) \wedge P(x))$
- **Some S(x) is not P(x)**
  - $\exists x\ (S(x) \wedge \neg P(x))$

Nested quantifiers

- More than one quantifier may be necessary to capture the meaning of a statement in the predicate logic.

Example:

- Every real number has its corresponding negative.
- **Translation:**
  - Assume:
    - a real number is denoted as x and its negative as y
    - A predicate P(x,y) denotes: "$x + y = 0$"
- Then we can write:

  $\forall x\ \exists\ y\ P(x,y)$

Nested quantifiers

- More than one quantifier may be necessary to capture the meaning of a statement in the predicate logic.

Example:

- There is a person who loves everybody.
- **Translation:**
  - Assume:
    - Variables x and y denote people
    - A predicate L(x,y) denotes: "x loves y"
- Then we can write in the predicate logic:

  $\exists x \forall y \; L(x,y)$

Order of quantifiers

The order of nested quantifiers matters if quantifiers are of different type

- $\forall x \exists y \; L(x,y)$ is not the same as $\exists y \forall x \; L(x,y)$

Example:

- Assume L(x,y) denotes "x loves y"

- Then: $\forall x \exists y \; L(x,y)$
- Translates to: Everybody loves somebody.
- And: $\exists y \forall x \; L(x,y)$
- Translates to: There is someone who is loved by everyone.

The meaning of the two is different.