Propositional logic: Equivalences
Predicate logic

Propositional logic: review

- **Propositional logic**: a formal language for making logical inferences
- A **proposition** is a statement that is either true or false.
- A **compound proposition** can be created from other propositions using logical connectives
- The **truth of a compound proposition** is defined by truth values of elementary propositions and the meaning of connectives.
- The **truth table for a compound proposition**: table with entries (rows) for all possible combinations of truth values of elementary propositions.
Tautology and Contradiction

Definitions:

• A compound proposition that is always true for all possible truth values of the propositions is called a tautology.
• A compound proposition that is always false is called a contradiction.
• A proposition that is neither a tautology nor contradiction is called a contingency.

Example:  \( p \lor \neg p \) is a tautology.

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<tr>
<th>( p )</th>
<th>( \neg p )</th>
<th>( p \lor \neg p )</th>
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Equivalence

• Some propositions may be equivalent. Their truth values in the truth table are the same.

• Example: \( p \rightarrow q \) is equivalent to \( \neg q \rightarrow \neg p \) (contrapositive)

<table>
<thead>
<tr>
<th>( p )</th>
<th>( q )</th>
<th>( p \rightarrow q )</th>
<th>( \neg q \rightarrow \neg p )</th>
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Equivalent statements are important for logical reasoning since they can be substituted and can help us to make a logical argument.
Logical equivalence

• **Definition:** The propositions $p$ and $q$ are called **logically equivalent** if $p \leftrightarrow q$ is a tautology (alternately, if they have the same truth table). The notation $p \leftrightarrow q$ denotes $p$ and $q$ are logically equivalent.

Examples of equivalences:

• **DeMorgan's Laws:**
  • 1) $\neg(p \lor q) \leftrightarrow \neg p \land \neg q$
  • 2) $\neg(p \land q) \leftrightarrow \neg p \lor \neg q$

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Equivalence

Example of important equivalences

• **DeMorgan's Laws:**
  • 1) $\neg(p \lor q) \leftrightarrow \neg p \land \neg q$
  • 2) $\neg(p \land q) \leftrightarrow \neg p \lor \neg q$

Use the truth table to prove that the two propositions are logically equivalent

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$\neg p$</th>
<th>$\neg q$</th>
<th>$\neg(p \lor q)$</th>
<th>$p \land \neg q$</th>
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</table>
Important logical equivalences

• Identity
  – \( p \land T \iff p \)
  – \( p \lor F \iff p \)

• Domination
  – \( p \lor T \iff T \)
  – \( p \land F \iff F \)

• Idempotent
  – \( p \lor p \iff p \)
  – \( p \land p \iff p \)

• Double negation
  – \( \neg \neg p \iff p \)

• Commutative
  – \( p \lor q \iff q \lor p \)
  – \( p \land q \iff q \land p \)

• Associative
  – \( (p \lor q) \lor r \iff p \lor (q \lor r) \)
  – \( (p \land q) \land r \iff p \land (q \land r) \)
Important logical equivalences

- **Distributive**
  - \( p \lor (q \land r) \iff (p \lor q) \land (p \lor r) \)
  - \( p \land (q \lor r) \iff (p \land q) \lor (p \land r) \)

- **De Morgan**
  - \( \neg (p \lor q) \iff \neg p \land \neg q \)
  - \( (p \land q) \iff \neg p \lor \neg q \)

- **Other useful equivalences**
  - \( p \lor \neg p \iff T \)
  - \( p \land \neg p \iff F \)
  - \( p \rightarrow q \iff (\neg p \lor q) \)

Using logical equivalences

- **Equivalences can be used in proofs.** A proposition or its part can be transformed using equivalences and some conclusion can be reached.

**Example:** Show that \((p \land q) \rightarrow p\) is a tautology.

- Proof: (we must show \((p \land q) \rightarrow p \iff T\))
  \[
  (p \land q) \rightarrow p \iff \neg(p \land q) \lor p \\
  \iff [\neg p \lor \neg q] \lor p \text{ DeMorgan} \\
  \iff [\neg q \lor \neg p] \lor p \text{ Commutative} \\
  \iff \neg q \lor [\neg p \lor p] \text{ Associative} \\
  \iff \neg q \lor [T] \text{ Useful} \\
  \iff T \text{ Domination}
  \]
Using logical equivalences

• **Equivalences can be used in proofs.** A proposition or its part can be transformed using equivalences and some conclusion can be reached.

**Example:** Show \((p \land q) \rightarrow p\) is a tautology.

• Alternative proof:

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<th>p \land q</th>
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• **Proofs that rely on logical equivalences can replace truth table approach**
  – Why?
  – The truth table has \(2^n\) rows, where \(n\) is the number of elementary propositions
  – If \(n\) is large building the truth table may become infeasible
Using logical equivalences

- Equivalences can be used in proofs. A proposition or its part can be transformed using equivalences and some conclusion can be reached.

- **Example 2:** Show \((p \rightarrow q) \iff (\neg q \rightarrow \neg p)\)

  **Proof:**
  
  \[
  (p \rightarrow q) \iff (\neg q \rightarrow \neg p)
  \]
  \[
  \iff \neg(\neg q) \lor (\neg p) \quad \text{Useful}
  \]
  \[
  \iff q \lor (\neg p) \quad \text{Double negation}
  \]
  \[
  \iff \neg p \lor q \quad \text{Commutative}
  \]
  \[
  \iff p \rightarrow q \quad \text{Useful}
  \]

End of proof

Limitations of the propositional logic

- **Propositional logic:** the world is described in terms of propositions
- A proposition is a statement that is either true or false.
- **Limitations:**
  - objects in elementary statements, their properties and relations are not explicitly represented in the propositional logic
- **Example:**
  - “John is a UPitt student.”

  \[
  \begin{array}{c}
  \text{John} \rightarrow \text{a UPitt student} \\
  \text{object} \rightarrow \text{a property}
  \end{array}
  \]

  - Objects and properties are hidden in the statement, it is not possible to reason about them
Limitations of the propositional logic

- **Statements for groups of objects**
  - In propositional logic these must be exhaustively enumerated

- **Example:**
  - If John is a CS UPitt graduate then John has passed cs441

**Translation:**
- John is a CS UPitt graduate $\rightarrow$ John has passed cs441

Similar statements can be written for other Upitt graduates:
- Ann is a CS Upitt graduate $\rightarrow$ Ann has passed cs441
- Ken is a CS Upitt graduate $\rightarrow$ Ken has passed cs441
- ...

- **What is a more natural solution to express the above knowledge?**

- **Solution:** make statements with **variables**
  - If $x$ is a CS UPitt graduate then $x$ has passed cs441
  - $x$ is a CS UPitt graduate $\rightarrow$ $x$ has passed cs441
Predicate logic

Remedies the limitations of the propositional logic
• Explicitly models objects and their properties
• Allows to make statements with variables and quantify them

Basic building blocks of the predicate logic:
• **Constant** – models a specific object
  Examples: “John”, “France”, “7”
• **Variable** – represents object of specific type (defined by the universe of discourse)
  Examples: x, y
  (universe of discourse can be people, students, numbers)
• **Predicate** - over one, two or many variables or constants.
  – Represents properties or relations among objects
  Examples: Red(car23), student(x), married(John,Ann)

Predicates

**Predicates** represent properties or relations among objects

A predicate P(x) assigns a value **true or false** to each x depending on whether the property holds or not for x.
• The assignment is best viewed as a big table with the variable x substituted for objects from the universe of discourse

**Example:**
• Assume **Student(x)** where the universe of discourse are people
  • Student(John) …. T  (if John is a student)
  • Student(Ann) …. T  (if Ann is a student)
  • Student(Jane) ..... F  (if Jane is not a student)
  • …
Predicates

Assume a predicate $P(x)$ that represents the statement:
• $x$ is a prime number

What are the truth values of:
• $P(2)$   T
• $P(3)$   T
• $P(4)$   F
• $P(5)$   T
• $P(6)$   F
• $P(7)$   T

All statements $P(2), P(3), P(4), P(5), P(6), P(7)$ are propositions

Predicates

Assume a predicate $P(x)$ that represents the statement:
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What are the truth values of:
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• $P(6)$   F
• $P(7)$   T

Is $P(x)$ a proposition? No. Many possible substitutions are possible.
Predicates

**Important:**
- predicate $P(x)$ is **not a proposition** since there are more objects it can be applied to

This is the same as in propositional logic …

**… But the difference is:**
- propositional logic does not let us go inside the statements and manipulate $x$
- predicate logic allows us to explicitly manipulate and substitute for the objects

Predicates

- Predicates can have **more arguments** which represent the relations between objects

**Example:**
- $\text{Older}(\text{John}, \text{Peter})$ denotes ‘John is older than Peter’
  - this is a proposition because it is either true or false
- $\text{Older}(x,y)$ - ‘$x$ is older than $y$’
  - not a proposition, but after the substitution it becomes one
Predicates

- Predicates can have **more arguments** which represent the **relations between objects**

**Example:**
- Let Q(x,y) denote ‘x+5 > y’
  - Is Q(x,y) a proposition? **No!**
  - Is Q(3,7) a proposition? **Yes. It is true.**
  - What is the truth value of
    - Q(3,7)  **T**
    - Q(1,6)  **F**
    - Q(2,2)  **T**
  - Is Q(3,y) a proposition? **No!** We cannot say if it is true or false.

Quantified statements

**Predicate logic allows us to make statements about groups of objects**
- To do this we use special quantified expressions

Two types of quantified statements:
- **universal**
  - **Example:** ‘all CS Upitt graduates have to pass cs441’
    - the statement is true for all graduates
- **existential**
  - **Example:** ‘Some CS Upitt students graduate with honor.’
    - the statement is true for some people
**Universal quantifier**

**Defn:** The universal quantification of \( P(x) \) is the proposition "\( P(x) \) is true for all values of \( x \) in the universe of discourse." The notation \( \forall x \ P(x) \) denotes the universal quantification of \( P(x) \), and is expressed as *for every \( x \), \( P(x) \).*

**Example:**
- Let \( P(x) \) denote \( x > x - 1 \).
- What is the truth value of \( \forall x \ P(x) \)?
- Assume the universe of discourse of \( x \) is all real numbers.
- **Answer:** Since every number \( x \) is greater than itself minus 1. Therefore, \( \forall x \ P(x) \) is true.

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**Universal quantifier**

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**Example 2:**
- Let \( T(x) \) denote \( x > 5 \).
- What is the truth value of \( \forall x \ T(x) \)?
- Assume the universe of discourse of \( x \) is all real numbers.
- **Answer:**
  - Since \( 3 > 5 \) is false. So, \( T(x) \) is not true for all values of \( x \). Therefore, it is **false that \( \forall x \ T(x) \).**
Universal quantifier

Quantification converts a propositional function into a proposition by binding a variable to a set of values from the universe of discourse.

Example:
- Let P(x) denote x > x - 1.
- Is P(x) a proposition? No. Many possible substitutions.
- Is ∀x P(x) a proposition? Yes. True if for all x from the universe of discourse P(x) is true. Which holds?

- Is ∀x Q(x,y) a proposition? No. The variable y is free and can be substituted by many objects.