CS 441 Discrete Mathematics for CS
Lecture 23

Relations III.

Milos Hauskrecht
milos@cs.pitt.edu
5329 Sennott Square

Closures on relations

- **Relations can have different properties:**
  - **reflexive,**
  - **symmetric**
  - **transitive**
- Because of that we can have:
  - **symmetric,**
  - **reflexive and**
  - **transitive**

  closures.

Closures

Definition: Let R be a relation on a set A. A relation S on A with property P is called **the closure of R with respect to P** if S is a subset of every relation Q ($S \subseteq Q$) with property P that contains R ($R \subseteq Q$).

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Example (symmetric closure):

- Assume R={(1,2),(1,3), (2,2)} on A={1,2,3}.
- What is the symmetric closure S of R?
- S=?

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Example (a symmetric closure):

- Assume R={(1,2),(1,3), (2,2)} on A={1,2,3}.
- What is the symmetric closure S of R?
- S = {(1,2),(1,3), (2,2)} ∪ {(2,1), (3,1)}
  = {(1,2),(1,3), (2,2),(2,1), (3,1)}

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Example (transitive closure):

- Assume R={(1,2), (2,2), (2,3)} on A={1,2,3}.
- **Is R transitive?**

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Example (transitive closure):

- Assume R={(1,2), (2,2), (2,3)} on A={1,2,3}.
- **Is R transitive? No.**
- **How to make it transitive?**
- **S = ?**

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Example (transitive closure):

- Assume R={(1,2), (2,2), (2,3)} on A={1,2,3}.
- **Is R transitive? No.**
- **How to make it transitive?**
- **S = {(1,2), (2,2), (2,3)} $\cup$ {(1,3)}**
  = {(1,2), (2,2), (2,3),(1,3)}
- S is the transitive closure of R

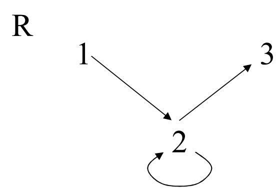
Transitive closure

We can represent the relation on the graph. Finding a transitive closure corresponds to finding all pairs of elements that are connected with a directed path (or digraph).

Example:

Assume R={(1,2), (2,2), (2,3)} on A={1,2,3}.

Transitive closure S = {(1,2), (2,2), (2,3),(1,3)}.



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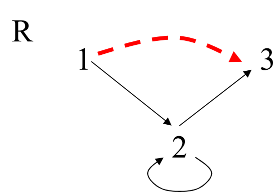

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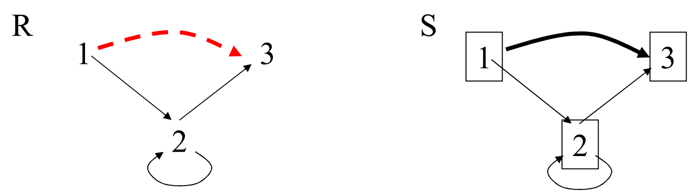
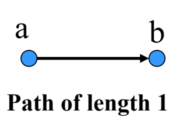
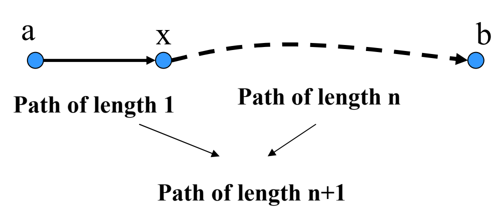


Transitive closure

Theorem: Let R be a relation on a set A. There is a path of length n from a to b if and only if $(a,b) \in R^n$.

Proof (math induction):





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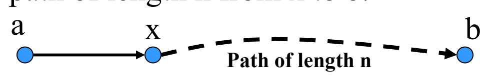
Proof (math induction):

- **P(1):** There is a path of length 1 from a to b if and only if $(a,b) \in R^1$, by the definition of R.
- **Show P(n) → P(n+1):** Assume there is a path of length n from a to b if and only if $(a,b) \in R^n \rightarrow$ there is a path of length n+1 from a to b if and only if $(a,b) \in R^{n+1}$.
- There is a path of length n+1 from a to b if and only if there exists an $x \in A$, such that $(a,x) \in R$ (a path of length 1) and $(x,b) \in R^n$ is a path of length n from x to b.



- $(x,b) \in R^n$ holds due to P(n). Therefore, there is a path of length n + 1 from a to b. This also implies that $(a,b) \in R^{n+1}$.

Connectivity relation

Definition: Let R be a relation on a set A. The **connectivity relation** R* consists of all pairs (a,b) such that there is a path (of any length, ie. 1 or 2 or 3 or ...) between a and b in R.

$$R^* = \bigcup_{k=1}^{\infty} R^k$$

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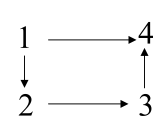

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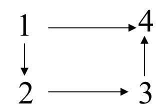

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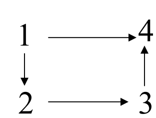

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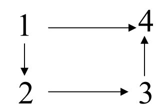

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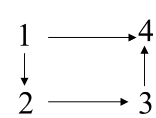

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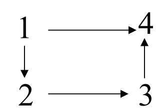

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- $R^4 = \varnothing$
- ...
- R* = {(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)}

Transitivity closure and connectivity relation

Theorem: The transitive closure of a relation R **equals** the connectivity relation R*.

Based on the following **Lemma.**

Lemma 1: Let A be a set with n elements, and R a relation on A. If there is a path from a to b, then there exists a path of length $< n$ in between (a,b). Consequently:

$$R^* = \bigcup_{k=1}^{n} R^k$$

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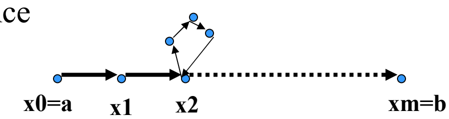
Proof (intuition):

- There are at most n different elements we can visit on a path if the path does not have loops

- Loops may increase the length but the same node is visited more than once



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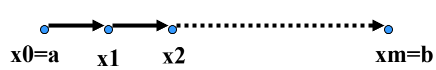

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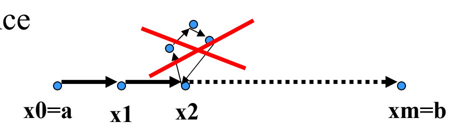

Equivalence relation

Definition: A relation R on a set A is called an **equivalence relation** if it is reflexive, symmetric and transitive.

Example: Let A = {0,1,2,3,4,5,6} and

- R= {(a,b)| a,b ∈ A, a ≡ b mod 3} (a is congruent to b modulo 3)

Congruencies:

- 0 mod 3 = ?

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- 0 mod 3 = 0   1 mod 3 = 1   2 mod 3 = 2   3 mod 3 = ?

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- 4 mod 3 = ?

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Relation R has the following pairs:

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Relation R has the following pairs:

- (0,0) (0,3), (3,0), (0,6), (6,0)
- (3,3), (3,6) (6,3), (6,6) (1,1),(1,4), (4,1), (4,4)
- (2,2), (2,5), (5,2), (5,5)

Equivalence relation

- **Relation R on A={0,1,2,3,4,5,6} has the following pairs:**
  (0,0) (0,3), (3,0), (0,6), (6,0)
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- Is R reflexive?

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- Is R reflexive? **Yes.**
- Is R symmetric?

Equivalence relation

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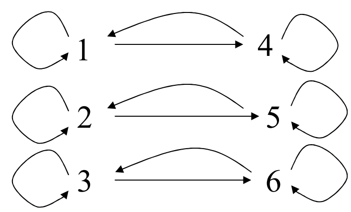
- Is R reflexive? **Yes.**
- Is R symmetric? **Yes.**
- Is R transitive?



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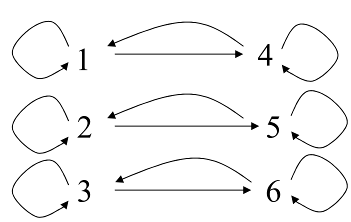
- Is R reflexive? **Yes.**
- Is R symmetric? **Yes.**
- Is R transitive. **Yes.**

Then
- **R is an equivalence relation**.



Equivalence class

Definition: Let R be an equivalence relation on a set A. The set { x ∈ A | a R x} is called **the equivalence class of a,** denoted by $[a]_R$ or simply [a] when there is only one relation R. If b ∈ [a] then b is called **a representative of this equivalence class**.

Example:

- Assume R={(a,b) | a ≡ b mod 3} for A={0,1,2,3,4,5,6}
- Pick an element a =0.
- $[0]_R = \{0,3,6\}$
- Element 1: $[1]_R = ?$

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- Assume $R=\{(a,b) \mid a \equiv b \bmod 3\}$ for $A=\{0,1,2,3,4,5,6\}$
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Equivalence class

Example:

- Assume R={(a,b) | a ≡ b mod 3} for A={0,1,2,3,4,5,6}

Three different equivalence classes all together:

- $[0]_R = [3]_R = [6]_R = \{0,3,6\}$
- $[1]_R = [4]_R = \{1,4\}$
- $[2]_R = [5]_R = \{2,5\}$

Partition of a set S

Definition: Let S be a set. A collection of nonempty subsets of S $A_1, A_2, \ldots, A_k$ is called **a partition of S** if:

- $A_i \cap A_j = \varnothing, \; i \neq j$ and $S = \bigcup_{i=1}^{k} A_i$

Example: Let S={1,2,3,4,5,6} and

- $A_1 = \{0,3,6\}$ $A_2 = \{1,4\}$ $A_3 = \{2,5\}$
- Is A_1, $A_{2,}$, A_3 a partition of S ?

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- $A_1 = \{0,3,6\}$ $A_2 = \{1,4\}$ $A_3 = \{2,5\}$
- Is $A_1, A_{2,,} A_3$ a partition of S ? **Yes.**
- Give a partition of S ?

Partition of a set S

Definition: Let S be a set. A collection of nonempty subsets of S $A_1, A_2, \ldots, A_k$ is called **a partition of S** if:

- $A_i \cap A_j = \emptyset, \; i \neq j$ and $S = \bigcup_{i=1}^{k} A_i$

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- Is $A_1, A_{2,,} A_3$ a partition of S ? **Yes.**
- Give a partition of S ?
- {0,2,4,6} {1,3,5}
- {0} {1,2} {3,4,5} {6}

Equivalence class

Theorem: Let R be an **equivalence relation** on a set A. The following statements are equivalent:

- i) a R b
- ii) [a] = [b]
- iii) $[a] \cap [b] \neq \emptyset$.

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Proof: (i) $\rightarrow$ (ii)

- Suppose **a R b**, i.e., $(a,b) \in R$. Want to show **[a] = [b].**
- Let $\mathbf{x} \in \mathbf{[a]} \rightarrow (a,x) \in R$.
- Since R is symmetric $(b,a) \in R$.
- Since R is transitive, $(b,a) \in R$ and $(a,x) \in R \rightarrow (b,x) \in R$. Thus, $x \in [b]$.
- Let $\mathbf{x} \in \mathbf{[b]} \rightarrow (b,x) \in R$.
- Since R is transitive, $(a,b) \in R$ and $(b,x) \in R \rightarrow (a,x) \in R$. Thus $x \in [a]$.
- Therefore [a] = [b].

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Proof: (ii) $\rightarrow$ (iii)

- Suppose [a] = [b]. Want to show $[a] \cap [b] \neq \emptyset$.
- Since R is reflexive, $a \in [a] \rightarrow [a] \neq \emptyset$ and the result follows.

Equivalence class

Theorem: Let R be an **equivalence relation** on a set A. The following statements are equivalent:

- i) a R b
- ii) [a] = [b]
- iii) $[a] \cap [b] \neq \emptyset$.

Proof: (iii) $\rightarrow$ (i)

- Suppose $[a] \cap [b] \neq \emptyset$, want to show a R b.
- $[a] \cap [b] \neq \emptyset \rightarrow x \in [a] \cap [b] \rightarrow x \in [a]$ and $x \in [b] \rightarrow (a,x)$ and $(b,x) \in R$.
- Since R is symmetric $(x,b) \in R$. By the transitivity of R $(a,x) \in R$ and $(x,b) \in R$ implies $(a,b) \in R \rightarrow$ a R b.