Relations II.

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Combining relations

**Definition**: Let A and B be sets. A binary relation from A to B is a subset of a Cartesian product A x B.

or \( R \subseteq A \times B \) means R is a set of ordered pairs of the form \((a,b)\) where \(a \in A\) and \(b \in B\).

**Combining Relations**

- Relations are sets \(\rightarrow\) combinations via set operations
- Set operations of: union, intersection, difference and symmetric difference.
Combining relations

**Example:**
- Let $A = \{1,2,3\}$ and $B = \{u,v\}$ and
- $R_1 = \{(1,u), (2,u), (2,v), (3,u)\}$
- $R_2 = \{(1,v),(3,u),(3,v)\}$

**What is:**
- $R_1 \cup R_2 = \{(1,u),(1,v),(2,u),(2,v),(3,u),(3,v)\}$
- $R_1 \cap R_2 = \{(3,u)\}$
- $R_1 - R_2 = \{(1,u),(2,u),(2,v)\}$
- $R_2 - R_1 = \{(1,v),(3,v)\}$

Combination of relations

- Can the relation formed by taking the union or intersection or composition of two relations $R_1$ and $R_2$ be represented in terms of matrix operations? **Yes**
### Union: matrix implementation

**Definition.** The **join**, denoted by $\lor$, of two $m$-by-$n$ matrices $(a_{ij})$ and $(b_{ij})$ of 0s and 1s is an $m$-by-$n$ matrix $(m_{ij})$ where

- $m_{ij} = a_{ij} \lor b_{ij}$ for all $i,j$
- = pairwise or (disjunction)

**Example:**

- Let $A = \{1,2,3\}$ and $B = \{u,v\}$ and
- $R_1 = \{(1,u), (2,u), (2,v), (3,u)\}$
- $R_2 = \{(1,v),(3,u),(3,v)\}$

<table>
<thead>
<tr>
<th></th>
<th>$R_1$</th>
<th>$R_2$</th>
<th>$R_1 \lor R_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_{R_1}$</td>
<td>\begin{bmatrix} 1 &amp; 0 \ 1 &amp; 1 \ 1 &amp; 0 \end{bmatrix}</td>
<td>\begin{bmatrix} 0 &amp; 1 \ 1 &amp; 1 \ 1 &amp; 0 \end{bmatrix}</td>
<td>\begin{bmatrix} 1 &amp; 1 \ 1 &amp; 1 \ 1 &amp; 1 \end{bmatrix}</td>
</tr>
</tbody>
</table>

### Intersection: matrix implementation

**Definition.** The **meet**, denoted by $\land$, of two $m$-by-$n$ matrices $(a_{ij})$ and $(b_{ij})$ of 0s and 1s is an $m$-by-$n$ matrix $(m_{ij})$ where

- $m_{ij} = a_{ij} \land b_{ij}$ for all $i,j$
- = pairwise and (conjunction)

**Example:**

- Let $A = \{1,2,3\}$ and $B = \{u,v\}$ and
- $R_1 = \{(1,u), (2,u), (2,v), (3,u)\}$
- $R_2 = \{(1,v),(3,u),(3,v)\}$

<table>
<thead>
<tr>
<th></th>
<th>$R_1$</th>
<th>$R_2$</th>
<th>$R_1 \land R_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_{R_1}$</td>
<td>\begin{bmatrix} 1 &amp; 0 \ 1 &amp; 1 \ 1 &amp; 0 \end{bmatrix}</td>
<td>\begin{bmatrix} 0 &amp; 1 \ 1 &amp; 1 \ 1 &amp; 0 \end{bmatrix}</td>
<td>\begin{bmatrix} 0 &amp; 0 \ 0 &amp; 0 \ 0 &amp; 0 \end{bmatrix}</td>
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Composite of relations

**Definition:** Let \( R \) be a relation from a set \( A \) to a set \( B \) and \( S \) a relation from \( B \) to a set \( C \). The **composite of \( R \) and \( S \)** is the relation consisting of the ordered pairs \((a,c)\) where \( a \in A \) and \( c \in C \), and for which there is a \( b \in B \) such that \((a,b)\) \( \in \) \( R \) and \((b,c)\) \( \in \) \( S \). We denote the composite of \( R \) and \( S \) by \( S \circ R \).

**Example:**

\[
\begin{array}{ccc}
A & B & C \\
a & b & c \\
\end{array}
\]

\( (a,c) \in S \circ R \)

\[
R \\
S
\]

Examples:
- Let \( A = \{1,2,3\} \), \( B = \{0,1,2\} \) and \( C = \{a,b\} \).
- \( R = \{(1,0), (1,2), (3,1),(3,2)\} \)
- \( S = \{(0,b),(1,a),(2,b)\} \)
- \( S \circ R = ? \)
Composite of relations

**Definition:** Let R be a relation from a set A to a set B and S a relation from B to a set C. The **composite of R and S** is the relation consisting of the ordered pairs (a,c) where a ∈ A and c ∈ C, and for which there is a b ∈ B such that (a,b) ∈ R and (b,c) ∈ S. We denote the composite of R and S by $S \circ R$.

**Example:**
- Let A = \{1,2,3\}, B = \{0,1,2\} and C = \{a,b\}.
- R = \{(1,0), (1,2), (3,1),(3,2)\}
- S = \{(0,b),(1,a),(2,b)\}
- $S \circ R = \{(1,b),(3,a),(3,b)\}$

Composite: matrix implementation

**Definition.** The **Boolean product**, denoted by $\Theta$, of an m-by-n matrix $(a_{ij})$ and n-by-p matrix $(b_{jk})$ of 0s and 1s is an m-by-p matrix $(m_{ik})$ where

$m_{ik} = \begin{cases} 1, & \text{if } a_{ij} = 1 \text{ and } b_{jk} = 1 \text{ for some } k=1,2,...,n \\ 0, & \text{otherwise} \end{cases}$

**Examples:**
- Let A = \{1,2,3\}, B = \{0,1,2\} and C = \{a,b\}.
- R = \{(1,0), (1,2), (3,1),(3,2)\}
- S = \{(0,b),(1,a),(2,b)\}
- $S \circ R = \{(1,b),(3,a),(3,b)\}$
Implementation of composite

Examples:

• Let A = \{1,2\}, \{1,2,3\} C = \{a,b\}
• R = \{(1,2),(1,3),(2,1)\} is a relation from A to B
• S = \{(1,a),(3,b),(3,a)\} is a relation from B to C.
• \(S \circ R = \{(1,b),(1,a),(2,a)\}\)

\[
\begin{array}{ccc}
0 & 1 & 1 \\
M_R = 1 & 0 & 0 \\
\end{array}
\begin{array}{ccc}
1 & 0 \\
M_S = 0 & 0 \\
\end{array}
\begin{array}{c}
1 \\
1 \\
\end{array}
\]

\(M_R \otimes M_S = ?\)

Implementation of composite

Examples:

• Let A = \{1,2\}, \{1,2,3\} C = \{a,b\}
• R = \{(1,2),(1,3),(2,1)\} is a relation from A to B
• S = \{(1,a),(3,b),(3,a)\} is a relation from B to C.
• \(S \circ R = \{(1,b),(1,a),(2,a)\}\)

\[
\begin{array}{ccc}
0 & 1 & 1 \\
M_R = 1 & 0 & 0 \\
\end{array}
\begin{array}{ccc}
1 & 0 \\
M_S = 0 & 0 \\
\end{array}
\begin{array}{c}
1 \\
1 \\
\end{array}
\]

\(M_R \otimes M_S = x \ x \\
\ x \ x\)
Implementation of composite

Examples:
- Let A = {1, 2}, {1, 2, 3} C = {a, b}
- R = {(1, 2), (1, 3), (2, 1)} is a relation from A to B
- S = {(1, a), (3, b), (3, a)} is a relation from B to C.
- S o R = {(1, b), (1, a), (2, a)}

\[
\begin{array}{ccc}
0 & 1 & 1 \\
M_R = 1 & 0 & 0 \\
\end{array}
\begin{array}{cc}
M_S = & 1 & 0 \\
& 0 & 0 \\
& 1 & 1 \\
\end{array}
\]

\[
M_R \odot M_S = \begin{array}{cc}
? & x \\
x & x \\
\end{array}
\]
### Implementation of composite

**Examples:**

- Let $A = \{1,2\}, \{1,2,3\}$, $C = \{a,b\}$
- $R = \{(1,2),(1,3),(2,1)\}$ is a relation from $A$ to $B$
- $S = \{(1,a),(3,b),(3,a)\}$ is a relation from $B$ to $C$.
- $S \circ R = \{(1,b),(1,a),(2,a)\}$

<table>
<thead>
<tr>
<th>$M_R$</th>
<th>$M_S$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 0 0</td>
<td>0 0 1</td>
</tr>
</tbody>
</table>

$M_R \odot M_S = \begin{cases} 1 & \text{if } x = x' \\ x & \text{otherwise} \end{cases}$
Examples:

- Let $A = \{1,2\}, \{1,2,3\}$ and $C = \{a,b\}$.
- $R = \{(1,2),(1,3),(2,1)\}$ is a relation from $A$ to $B$.
- $S = \{(1,a),(3,b),(3,a)\}$ is a relation from $B$ to $C$.
- $S \circ R = \{(1,b),(1,a),(2,a)\}$

\[
\begin{array}{cccc}
0 & 1 & 1 & 1 & 0 \\
M_R &=& 1 & 0 & 0 & M_S &=& 0 & 0 \\
\end{array}
\]

\[
\begin{array}{cccc}
M_R \odot M_S &=& 1 & 1 \\
& & 1 & x \\
\end{array}
\]

$M_{S \circ R} = ?$
Implementation of composite

**Examples:**
- Let \( A = \{1,2\} \), \( B = \{1,2,3\} \), \( C = \{a,b\} \)
- \( R = \{(1,2),(1,3),(2,1)\} \) is a relation from \( A \) to \( B \)
- \( S = \{(1,a),(3,b),(3,a)\} \) is a relation from \( B \) to \( C \).
- \( S \circ R = \{(1,b),(1,a),(2,a)\} \)

\[
\begin{array}{ccc}
0 & 1 & 1 \\
1 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 1 \\
\end{array}
\]

\[
M_R = \begin{array}{cc}
1 & 0 \\
0 & 1 \\
\end{array}, \quad M_S = \begin{array}{cc}
0 & 0 \\
1 & 1 \\
\end{array}
\]

\[
M_R \odot M_S = \begin{array}{cc}
1 & 1 \\
1 & 0 \\
\end{array}
\]

\[
M_{S \circ R} = \begin{array}{cc}
1 & 1 \\
1 & 0 \\
\end{array}
\]

Composite of relations

**Definition:** Let \( R \) be a relation on a set \( A \). The **powers** \( R^n \), \( n = 1,2,3,... \) is defined inductively by

- \( R^1 = R \) and \( R^{n+1} = R^n \circ R \).

**Examples**
- \( R = \{(1,2),(2,3),(2,4),(3,3)\} \) is a relation on \( A = \{1,2,3,4\} \).
- \( R^1 = R = \{(1,2),(2,3),(2,4),(3,3)\} \)
- \( R^2 = \{(1,3),(1,4),(2,3),(3,3)\} \)
- \( R^3 = \{(1,3),(2,3),(3,3)\} \)
- \( R^4 = \{(1,3),(2,3),(3,3)\} \)
- \( R^k = R^3, \quad k > 3. \)
Representing binary relations with graphs

- We can graphically represent a binary relation $R$ from $A$ to $B$ as follows:
  - if $a R b$ then draw an arrow from $a$ to $b$.

Example:
- Relation $R_{\text{div}}$ (from previous lectures) on $A=\{1,2,3,4\}$
- $R_{\text{div}} = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$

Representing relations on a set with digraphs

Definition: A directed graph or digraph consists of a set of vertices (or nodes) together with a set $E$ of ordered pairs of elements of $V$ called edges (or arcs). The vertex $a$ is called the initial vertex of the edge $(a,b)$ and vertex $b$ is the terminal vertex of this edge. An edge of the form $(a,a)$ is called a loop.

Example
- Relation $R_{\text{div}} = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$
Composite of relations

Definition: Let R be a relation on a set A. The powers $R^n$, $n = 1, 2, 3, \ldots$ is defined inductively by

- $R^1 = R$ and $R^{n+1} = R^n \circ R$.

Examples
- $R = \{(1,2), (2,3), (2,4), (3,3)\}$ is a relation on $A = \{1, 2, 3, 4\}$.

1 2 3 4

$R^1 = R = \{(1,2), (2,3), (2,4), (3,3)\}$

$R^2 = \{(1,3), (1,4), (2,3), (3,3)\}$

What does $R^2$ represent?
Composite of relations

**Definition**: Let $R$ be a relation on a set $A$. The powers $R^n$, $n = 1,2,3,...$ is defined inductively by

- $R^1 = R$ and $R^{n+1} = R^n \circ R$.

**Examples**

- $R = \{(1,2), (2,3), (2,4), (3,3)\}$ is a relation on $A = \{1,2,3,4\}$.

- $R^1 = R = \{(1,2), (2,3), (2,4), (3,3)\}$
- $R^2 = \{(1,3), (1,4), (2,3), (3,3)\}$
- What does $R^2$ represent?
- Paths of length 2
Composite of relations

**Definition:** Let R be a relation on a set A. The powers $R^n$, $n = 1,2,3,...$ is defined inductively by

- $R^1 = R$ and $R^{n+1} = R^n \circ R$.

**Examples**

- $R = \{(1,2),(2,3),(2,4), (3,3)\}$ is a relation on $A = \{1,2,3,4\}$.

- $R^1 = R = \{(1,2),(2,3),(2,4), (3,3)\}$
- $R^2 = \{(1,3), (1,4), (2,3), (3,3)\}$
- What does $R^2$ represent?
- Paths of length 2
- $R^3 = \{(1,3), (2,3), (3,3)\}$ path of length 3

Transitive relation

**Definition (transitive relation):** A relation $R$ on a set $A$ is called **transitive** if

- $[(a,b) \in R \text{ and } (b,c) \in R] \rightarrow (a,c) \in R$ for all $a, b, c \in A$.

**Example 1:**

- $R_{\text{div}} = \{(a, b), \text{ if } a \mid b\}$ on $A = \{1,2,3,4\}$
- $R_{\text{div}} = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$
- Is $R_{\text{div}}$ transitive?
- Answer: Yes.
Connection to $R^n$

**Theorem:** The relation $R$ on a set $A$ is transitive if and only if $R^n \subseteq R$ for $n = 1,2,3,...$.

**Proof: bi-conditional (if and only if)**

($\leftarrow$) Suppose $R^n \subseteq R$, for $n =1,2,3,...$.

- Let $(a,b) \in R$ and $(b,c) \in R$.
- by the definition of $R \circ R$, $(a,c) \in R \circ R \subseteq R$.
- $R$ is transitive.

($\rightarrow$) Suppose $R$ is transitive. Show $R^n \subseteq R$, for $n =1,2,3,...$.

- Let $P(n) : R^n \subseteq R$. Math induction.
- **Basis Step:** $P(1)$ says $R^1 = R$ so, $R^1 \subseteq R$ is true.
- **Inductive Step:** show $P(n) \rightarrow P(n+1)$
- Want to show if $R^n \subseteq R$ then $R^{n+1} \subseteq R$.
- Let $(a,b) \in R^{n+1}$ then by the definition of $R^{n+1} = R^n \circ R$ there is an element $x \in A$ so that $(a,x) \in R$ and $(x,b) \in R^n \subseteq R$ (inductive hypothesis). In addition to $(a,x) \in R$ and $(x,b) \in R$, $R$ is transitive; so $(a,b) \in R$.
- Therefore, $R^{n+1} \subseteq R$. 
Number of reflexive relations

**Theorem:** The number of reflexive relations on a set $A$, where $|A| = n$ is: $2^{n(n-1)}$.

**Proof:**
- A reflexive relation $R$ on $A$ **must contain** all pairs $(a,a)$ where $a \in A$.
- All other pairs in $R$ are of the form $(a,b)$, $a \neq b$, such that $a, b \in A$.
- How many of these pairs are there? **Answer:** $n(n-1)$.
- How many subsets on $n(n-1)$ elements are there? **Answer:** $2^{n(n-1)}$.

Closures of relations

- Let $R=\{(1,1),(1,2),(2,1),(3,2)\}$ on $A = \{1 \ 2 \ 3\}$.
- Is this relation reflexive?
- **Answer:** No. Why?
- *(2,2) and (3,3) is not in $R$.*

- The question is what is the **minimal relation** $S \supseteq R$ that is reflexive?
- How to make $R$ reflexive with minimum number of additions?
- **Answer:** Add (2,2) and (3,3)
  - Then $S=\{(1,1),(1,2),(2,1),(3,2),(2,2),(3,3)\}$
  - $R \subseteq S$
  - The minimal set $S \supseteq R$ is called the **reflexive closure of $R$**.
Reflexive closure

The set $S$ is called the reflexive closure of $R$ if it:

- contains $R$
- has reflexive property
- is contained in every reflexive relation $Q$ that contains $R$ ($R \subseteq Q$), that is $S \subseteq Q$

Closures on relations

- Relations can have different properties:
  - reflexive,
  - symmetric
  - transitive

- Because of that we can have:
  - symmetric,
  - reflexive and
  - transitive closures.
Closures

Definition: Let R be a relation on a set A. A relation S on A with property P is called the closure of R with respect to P if S is a subset of every relation Q (S ⊆ Q) with property P that contains R (R ⊆ Q).

Example (a symmetric closure):
• Assume R={(1,2),(1,3), (2,2)} on A={1,2,3}.
• What is the symmetric closure S of R?
• S = {(1,2),(1,3), (2,2)} ∪ {2,1}, (3,1)}
  = {(1,2),(1,3), (2,2),(2,1), (3,1)}
Closures

Definition: Let R be a relation on a set A. A relation S on A with property P is called the closure of R with respect to P if S is a subset of every relation Q (S ⊆ Q) with property P that contains R (R ⊆ Q).

Example (transitive closure):
• Assume R={(1,2), (2,2), (2,3)} on A={1,2,3}.
• Is R transitive? No.
• How to make it transitive?
• S = {(1,2), (2,2), (2,3)} ∪ {(1,3)}
  = {(1,2), (2,2), (2,3), (1,3)}
• S is the transitive closure of R