CS 441 Discrete Mathematics for CS
Lecture 22

Relations II.

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Combining relations

Definition: Let A and B be sets. A **binary relation from A to B** is **a subset** of a Cartesian product A x B.

or $R \subseteq A \times B$ means R is a set of ordered pairs of the form (a,b) where $a \in A$ and $b \in B$.

Combining Relations

- **Relations are sets → combinations via set operations**
- Set operations of: **union, intersection, difference and symmetric difference.**

Combining relations

Example:

- Let A = {1,2,3} and B = {u,v} and
- R1 = {(1,u), (2,u), (2,v), (3,u)}
- R2 = {(1,v),(3,u),(3,v)}

What is:

- R1 ∪ R2 = {(1,u),(1,v),(2,u),(2,v),(3,u),(3,v)}
- R1 ∩ R2 = {(3,u)}
- R1 - R2 = {(1,u),(2,u),(2,v)}
- R2 - R1 = {(1,v),(3,v)}

Combination of relations

- Can the relation formed by taking the union or intersection or composition of two relations R1 and R2 be represented in terms of matrix operations? **Yes**

Union: matrix implementation

Definition. The **join,** denoted by $\vee$, of two m-by-n matrices (a_{ij}) and (b_{ij}) of 0s and 1s is an m-by-n matrix (m_{ij}) where

- $m_{ij} = a_{ij} \vee b_{ij}$ for all i,j

  = pairwise or (disjunction)

- **Example**:
- Let A = {1,2,3} and B = {u,v} and
- R1 = {(1,u), (2,u), (2,v), (3,u)}
- R2 = {(1,v),(3,u),(3,v)}
- **MR1 =**

1	**0**
1	**1**
1	**0**

MR2 =

0	**1**
0	**0**
1	**1**

M(R1 $\vee$ R2)=

1	**1**
1	**1**
1	**1**

Intersection: matrix implementation

Definition. The **meet,** denoted by $\wedge$, of two m-by-n matrices (a_{ij}) and (b_{ij}) of 0s and 1s is an m-by-n matrix (m_{ij}) where

- $m_{ij} = a_{ij} \wedge b_{ij}$ for all i,j

  = **pairwise and (conjunction)**

- **Example**:
- Let A = {1,2,3} and B = {u,v} and
- R1 = {(1,u), (2,u), (2,v), (3,u)}
- R2 = {(1,v),(3,u),(3,v)}
- **MR1 =**

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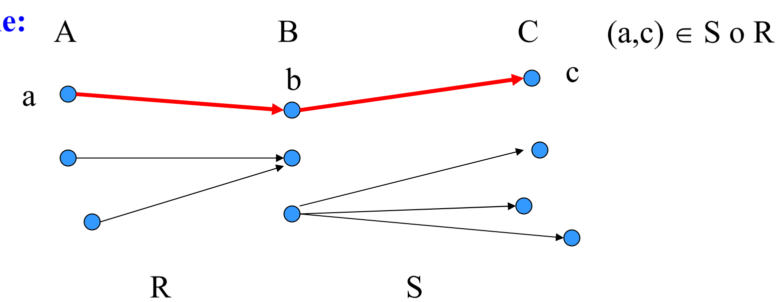
MR1 $\wedge$ MR2=

0	**0**
0	**0**
1	**0**

Composite of relations

Definition: Let R be a relation from a set A to a set B and S a relation from B to a set C. The **composite of R and S** is the relation consisting of the ordered pairs (a,c) where a ∈ A and c ∈ C, and for which there is a b ∈ B such that (a,b) ∈ R and (b,c) ∈ S. We denote the composite of R and S by S o R.

Example:



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Examples:

- Let A = {1,2,3}, B = {0,1,2} and C = {a,b}.
- R = {(1,0), (1,2), (3,1),(3,2)}
- S = {(0,b),(1,a),(2,b)}

- S o R = ?

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Composite: matrix implementation

Definition. The **Boolean product**, **denoted by ⊙,** of an m-by-n matrix (a_{ij}) and n-by-p matrix (b_{jk}) of 0s and 1s is an m-by-p matrix (m_{ik}) where

- m_{ik} = 1, if a_{ij} = 1 and b_{jk} = 1 for some k=1,2,...,n
  0, otherwise

Examples:

- Let A = {1,2,3}, B = {0,1,2} and C = {a,b}.
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Implementation of composite

Examples:

- Let A = {1,2}, {1,2,3} C = {a,b}
- R = {(1,2),(1,3),(2,1)} is a relation from A to B
- S = {(1,a),(3,b),(3,a)} is a relation from B to C.
- S o R = {(1,b),(1,a),(2,a)}

$$M_R = \begin{matrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{matrix} \qquad M_S = \begin{matrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{matrix}$$

$$M_R \odot M_S = ?$$

Implementation of composite

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$M_{S \circ R} =$?

Implementation of composite

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$$M_{S\,o\,R} = \begin{matrix} 1 & 1 \\ 1 & 0 \end{matrix}$$

Composite of relations

Definition: Let R be a relation on a set A. The **powers R^n**, n = 1,2,3,... is defined inductively by

- $\mathbf{R^1 = R}$ and $\mathbf{R^{n+1} = R^n \text{ o } R}$.

Examples

- R = {(1,2),(2,3),(2,4), (3,3)} is a relation on A = {1,2,3,4}.
- R^1 = R = {(1,2),(2,3),(2,4), (3,3)}
- R^2 = {(1,3), (1,4), (2,3), (3,3)}
- R^3 = {(1,3), (2,3), (3,3)}
- R^4 = {(1,3), (2,3), (3,3)}
- $R^k = R^3$, k > 3.

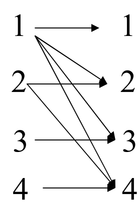
Representing binary relations with graphs

- We can graphically represent a binary relation R from A to B as follows:
  - if **a R b** then draw an arrow from a to b.

a → b

Example:

- Relation R_{div} (from previous lectures) on A={1,2,3,4}
- R_{div} = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)}

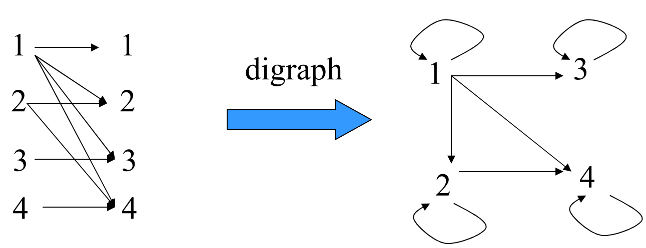


Representing relations on a set with digraphs

Definition: A **directed graph or digraph** consists of a set of vertices (or nodes) together with a set E of ordered pairs of elements of V called edges (or arcs). The vertex a is called the initial vertex of the edge (a,b) and vertex b is the terminal vertex of this edge. An edge of the form (a,a) is called a loop.

Example

- Relation R_{div} ={(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)}



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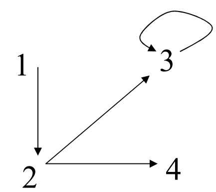

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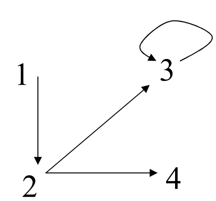

- $R^1 = R$ = {(1,2),(2,3),(2,4), (3,3)}
- R^2 = {(1,3), (1,4), (2,3), (3,3)}
- What does R^2 represent?

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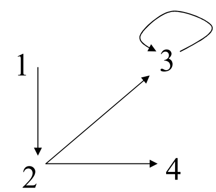

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- What does R^2 represent?
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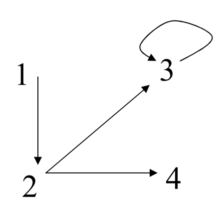

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- What does R^2 represent?
- Paths of length 2
- R^3 = {(1,3), (2,3), (3,3)}

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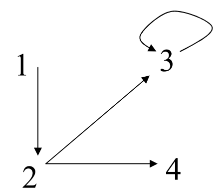

- R^1 = R = {(1,2),(2,3),(2,4), (3,3)}
- R^2 = {(1,3), (1,4), (2,3), (3,3)}
- What does R^2 represent?
- Paths of length 2
- R^3 = {(1,3), (2,3), (3,3)} path of length 3

Transitive relation

Definition (transitive relation): A relation R on a set A is called **transitive** if

- [(a,b) ∈ R and (b,c) ∈ R] → (a,c) ∈ R for all a, b, c ∈ A.

- **Example 1:**
- R_{div} ={(a b), if a |b} on A = {1,2,3,4}
- R_{div} = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)}
- **Is R_{div} transitive?**
- **Answer: Yes.**

Connection to R^n

Theorem: The relation R on a set A is transitive **if and only if** $R^n \subseteq R$ for n = 1,2,3,... .

Proof: bi-conditional (if and only if)

($\leftarrow$) Suppose $R^n \subseteq R$, for n =1,2,3,... .

- Let $(a,b) \in R$ and $(b,c) \in R$
- by the definition of R o R, $(a,c) \in R \circ R \subseteq R \rightarrow$
- R is transitive.

Connection to R^n

Theorem: The relation R on a set A is transitive **if and only if** $R^n \subseteq R$ for n = 1,2,3,... .

Proof: biconditional (if and only if)

($\rightarrow$) Suppose R is transitive. Show $R^n \subseteq R$, for n =1,2,3,... .

- Let P(n) : $R^n \subseteq R$. Math induction.
- **Basis Step:** P(1) says $R^1 = R$ so, $R^1 \subseteq R$ is true.
- **Inductive Step:** show $P(n) \rightarrow P(n+1)$
- Want to show if $R^n \subseteq R$ then $R^{n+1} \subseteq R$.
- Let $(a,b) \in R^{n+1}$ then by the definition of $R^{n+1} = R^n \circ R$ there is an element $x \in A$ so that $(a,x) \in R$ and $(x,b) \in R^n \subseteq R$ (inductive hypothesis). In addition to $(a,x) \in R$ and $(x,b) \in R$, R is transitive; so $(a,b) \in R$.
- Therefore, $R^{n+1} \subseteq R$.

Number of reflexive relations

Theorem: The number of reflexive relations on a set A, where | A | = n is: $2^{n(n-1)}$.

Proof:

- A reflexive relation R on A **must contain** all pairs (a,a) where $a \in A$.
- All other pairs in R are of the form (a,b), $a \neq b$, such that $a, b \in A$.
- How many of these pairs are there? Answer: n(n-1).
- How many subsets on n(n-1) elements are there?
- **Answer:** $2^{n(n-1)}$.

Closures of relations

- Let R={(1,1),(1,2),(2,1),(3,2)} on A ={1 2 3}.
- Is this relation reflexive?
- Answer: **No.** Why?
- **(2,2) and (3,3) is not in R.**

- The question is what is **the minimal relation** $S \supseteq R$ that is reflexive?
- How to make R reflexive with minimum number of additions?
- **Answer:** Add (2,2) and (3,3)
  - Then S= {(1,1),(1,2),(2,1),(3,2),(2,2),(3,3)}
  - $R \subseteq S$
  - The minimal set $S \supseteq R$ is called **the reflexive closure of R**

Reflexive closure

The set S is called **the reflexive closure of R** if it:

– contains R

– has reflexive property

– is contained in every reflexive relation Q that contains R (R $\subseteq$ Q) , that is S $\subseteq$ Q

Closures on relations

- **Relations can have different properties:**
  - **reflexive,**
  - **symmetric**
  - **transitive**
- Because of that we can have:
  - **symmetric,**
  - **reflexive and**
  - **transitive**

  closures.

Closures

Definition: Let R be a relation on a set A. A relation S on A with property P is called **the closure of R with respect to P** if S is a subset of every relation Q ($S \subseteq Q$) with property P that contains R ($R \subseteq Q$).

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Example (a symmetric closure):

- Assume R={(1,2),(1,3), (2,2)} on A={1,2,3}.
- What is the symmetric closure S of R?
- $S = \{(1,2),(1,3), (2,2)\} \cup \{(2,1), (3,1)\}$
  $= \{(1,2),(1,3), (2,2),(2,1), (3,1)\}$

Closures

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Example (transitive closure):

- Assume R={(1,2), (2,2), (2,3)} on A={1,2,3}.
- **Is R transitive? No.**
- **How to make it transitive?**
- **S = {(1,2), (2,2), (2,3)} $\cup$ {(1,3)}**
  = {(1,2), (2,2), (2,3),(1,3)}
- S is the transitive closure of R