CS 441 Discrete Mathematics for CS
Lecture 21

Relations

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Cartesian product (review)

- Let $A=\{a_1, a_2, ..a_k\}$ and $B=\{b_1,b_2,..b_m\}$.
- **The Cartesian product** A x B is defined by a set of pairs $\{(a_1\ b_1), (a_1, b_2), \ldots (a_1, b_m), \ldots, (a_k,b_m)\}$.

Example:

Let A={a,b,c} and B={1 2 3}. What is AxB?

Cartesian product (review)

- Let $A=\{a_1, a_2, ..a_k\}$ and $B=\{b_1,b_2,..b_m\}$.
- **The Cartesian product** A x B is defined by a set of pairs $\{(a_1\ b_1), (a_1, b_2), \ldots (a_1, b_m), \ldots, (a_k,b_m)\}$.

Example:

Let A={a,b,c} and B={1 2 3}. What is AxB?

AxB = {(a,1),(a,2),(a,3),(b,1),(b,2),(b,3)}

Binary relation

<u>Definition:</u> Let A and B be sets. A **binary relation from A to B** is **a subset of a Cartesian product A x B**.

Example: Let A={a,b,c} and B={1,2,3}.

- R={(a,1),(b,2),(c,2)} is an example of a relation from A to B.

Binary relation

Definition: Let A and B be sets. A **binary relation from A to B** is **a subset of a Cartesian product A x B**.

Example: Let A={a,b,c} and B={1,2,3}.

- R={(a,1),(b,2),(c,2)} is an example of a relation from A to B.
- Another example of a relation from A to B?

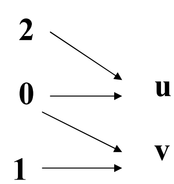
Representing binary relations

- We can graphically represent a binary relation R as follows:
  - if **a R b** then draw an arrow from a to b.

a → b

Example:

- Let A = {0, 1, 2}, B = {u,v} and R = { (0,u), (0,v), (1,v), (2,u) }
- Note: $R \subseteq A \times B$.
- **Graph:**



Representing binary relations

- We can represent a binary relation R by a **table** showing (marking) the ordered pairs of R.

Example:

- Let A = {0, 1, 2}, B = {u,v} and R = { (0,u), (0,v), (1,v), (2,u) }
- **Table:**

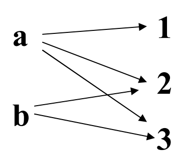
R	u	v
0	x	x
1		x
2	x	

or

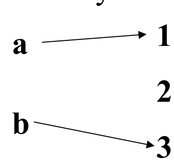
R	u	v
0	1	1
1	0	1
2	1	0

Relations and functions

- Relations represent **one to many relationships** between elements in A and B.
- **Example:**



- What is the difference between a **relation and a function from A to B**? A function on sets A,B A → B assigns to each element in the domain set A exactly one element from B. So it is **a special relation.**



Relation on the set

Definition: **A relation on the set A** is a relation from A to itself.

Example 1:

- Let A = {1,2,3,4} and R_{div} = {(a,b)| a divides b}
- What does R_{div} consist of?
- R_{div} = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)}
-

R	1	2	3	4
1	x	x	x	x
2		x		x
3			x	
4				x

Relation on the set

Example 2:

- Let A = {1,2,3,4}.
- Define a $R_{\neq}$ b if and only if a $\neq$ b.

$R_{\neq}$={(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,4),(4,1),(4,2),(4,3)}

-

R	1	2	3	4
1		x	x	x
2	x		x	x
3	x	x		x
4	x	x	x	

Relation on the set

Definition: **A relation on the set A** is a relation from A to itself.

Example 3:

- Let A = {1,2,3,4} and
- R_{fun} on A = {1,2,3,4} is defined as:
  - R_{fun} = {(1,2),(2,2),(3,3)}.

Binary relations

- **Theorem:** The number of binary relations on a set A, where | A | = n is:

$$2^{n^2}$$

- **Proof:**
- If | A | = n then the cardinality of the Cartesian product | A x A | = n^2.
- R is a binary relation on A if R ⊆ A x A (that is, R is a subset of A x A).
- The number of subsets of a set with k elements : 2^k
- The number of subsets of A x A is : $2^{|AxA|} = 2^{n^2}$

Binary relations

- **Example**: Let A = {1,2}
- What is A x A = **{(1,1),(1,2),(2,1),(2,2)}**
- **List of possible relations (subsets of A x A):**
- ∅ …. 1
- {(1,1)} {(1,2)} {(2,1)} {(2,2)} …. 4
- {(1,1), (1,2)} {(1,1),(2,1)} {(1,1),(2,2)} …. 6
  {(1,2),(2,1)} {(1,2),(2,2)} {(2,1),(2,2)}
- {(1,1),(1,2),(2,1)} {(1,1),(1,2),(2,2)} …. 4
  {(1,1),(2,1),(2,2)} {(1,2),(2,1),(2,2)}
- {(1,1),(1,2),(2,1),(2,2)} …. 1

16

- Use formula: $\mathbf{2^4 = 16}$

Properties of relations

Definition (reflexive relation) : A relation R on a set A is called **reflexive** if (a,a) ∈ R for every element a ∈ A.

Example 1:

- Assume relation R_{div} ={(a b), if a |b} on A = {1,2,3,4}
- **Is R_{div} reflexive?**
- R_{div} = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)}
- **Answer:** Yes. (1,1), (2,2), (3,3), and (4,4) ∈ A.

Reflexive relation

Reflexive relation

- R_{div} ={(a b), if a |b} on A = {1,2,3,4}
- R_{div} = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)}

$$MR_{div} = \begin{matrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}$$

- **A relation R is reflexive** if and only if MR has 1 in every position on its main diagonal.

Properties of relations

Definition (reflexive relation) : A relation R on a set A is called **reflexive** if (a,a) ∈ R for every element a ∈ A.

Example 2:

- Relation R_{fun} on A = {1,2,3,4} defined as:
  - R_{fun} = {(1,2),(2,2),(3,3)}.
- **Is R_{fun} reflexive?**
- **No.** It is not reflexive since (1,1) ∉ R_{fun}.

Properties of relations

Definition (irreflexive relation): A relation R on a set A is called **irreflexive** if **(a,a) $\notin$ R for every a $\in$ A.**

Example 1:

- Assume relation $R_{\neq}$ on A={1,2,3,4}, such that **a $R_{\neq}$ b** if and only if a $\neq$ b.
- **Is $R_{\neq}$ irreflexive?**
- **$R_{\neq}$={(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,4),(4,1),(4,2),(4,3)}**
- **Answer:** Yes. Because (1,1),(2,2),(3,3) and (4,4) $\notin$ $R_{\neq}$

Irreflexive relation

Irreflexive relation

- $R_{\neq}$ on A={1,2,3,4}, such that **a $R_{\neq}$ b** if and only if a $\neq$ b.
- **$R_{\neq}$={(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,4),(4,1),(4,2),(4,3)}**

$$MR = \begin{matrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{matrix}$$

- **A relation R is irreflexive** if and only if MR has 0 in every position on its main diagonal.

Properties of relations

Definition (irreflexive relation): A relation R on a set A is called **irreflexive** if **(a,a) $\notin$ R for every a $\in$ A.**

Example 2:

- R_{fun} on A = {1,2,3,4} defined as:
  - R_{fun} = {(1,2),(2,2),(3,3)}.
- **Is R_{fun} irreflexive?**
- **Answer: No**. Because (2,2) and (3,3) $\in$ R_{fun}

Properties of relations

Definition (symmetric relation): A relation R on a set A is called **symmetric** if

$\forall$ a, b $\in$ A (a,b) $\in$ R $\rightarrow$ (b,a) $\in$ R.

Example 1:

- R_{div} ={(a b), if a |b} on A = {1,2,3,4}
- **Is R_{div} symmetric?**
- R_{div} = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)}
- **Answer: No.** It is not symmetric since (1,2) $\in$ R but (2,1) $\notin$ R.

Properties of relations

Definition (symmetric relation): A relation R on a set A is called **symmetric** if

$\forall$ a, b $\in$ A (a,b) $\in$ R $\rightarrow$ (b,a) $\in$ R.

Example 2:

- $R_{\neq}$ on A={1,2,3,4}, such that **a $R_{\neq}$ b** if and only if a $\neq$ b.
- **Is $R_{\neq}$ symmetric ?**
- **$R_{\neq}$={(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,4),(4,1),(4,2),(4,3)}**
- **Answer: Yes. If** (a,b) $\in R_{\neq} \rightarrow$ (b,a) $\in R_{\neq}$

Symmetric relation

Symmetric relation:

- $R_{\neq}$ on A={1,2,3,4}, such that **a $R_{\neq}$ b** if and only if a $\neq$ b.
- **$R_{\neq}$={(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,4),(4,1),(4,2),(4,3)}**

$$MR = \begin{matrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{matrix}$$

- **A relation R is symmetric** if and only if $m_{ij} = m_{ji}$ for all i,j.

Properties of relations

- **Definition (antisymmetric relation)**: A relation on a set A is called **antisymmetric** if
  - $[(a,b) \in R \text{ and } (b,a) \in R] \rightarrow a = b$ where $a, b \in A$.

Example 1:

- Relation R_{fun} on A = {1,2,3,4} defined as:
  - $R_{fun} = \{(1,2),(2,2),(3,3)\}$.
- **Is R_{fun} antisymmetric?**
- **Answer: Yes.** It is antisymmetric

Antisymmetric relations

Antisymmetric relation

- relation $R_{fun} = \{(1,2),(2,2),(3,3)\}$

$$MR_{fun} = \begin{matrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}$$

- A relation is **antisymmetric** if and only if $m_{ij} = 1 \rightarrow m_{ij} = 0$ for $i \neq j$.

Properties of relations

Definition (antisymmetric relation): A relation on a set A is called **antisymmetric** if

- [(a,b) ∈ R and (b,a) ∈ R] → a = b where a, b ∈ A.

Example 2:

- $R_{\neq}$ on A={1,2,3,4}, such that **a $R_{\neq}$ b** if and only if a ≠ b.
- **Is $R_{\neq}$ antisymmetric ?**
- **$R_{\neq}$={(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,4),(4,1),(4,2),(4,3)}**
- **Answer: No.** It is not antisymmetric since (1,2) ∈ R and (2,1) ∈ R but 1≠ 2.

Properties of relations

Definition (transitive relation): A relation R on a set A is called **transitive** if

- [(a,b) ∈ R and (b,c) ∈ R] → (a,c) ∈ R for all a, b, c ∈ A.

- **Example 1:**
- R_{div} ={(a b), if a |b} on A = {1,2,3,4}
- R_{div} = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)}
- **Is R_{div} transitive?**
- **Answer: Yes.**

Properties of relations

Definition (transitive relation): A relation R on a set A is called **transitive** if

- $[(a,b) \in R \text{ and } (b,c) \in R] \rightarrow (a,c) \in R$ for all $a, b, c \in A$.

- **Example 2**:
- $R_{\neq}$ on A={1,2,3,4}, such that **a $R_{\neq}$ b** if and only if $a \neq b$.
- **$R_{\neq}$={(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,4),(4,1),(4,2),(4,3)}**
- **Is $R_{\neq}$ transitive?**
- **Answer: No.** It is not transitive since $(1,2) \in R$ and $(2,1) \in R$ but (1,1) is not an element of R.

Properties of relations

Definition (transitive relation): A relation R on a set A is called **transitive** if

- $[(a,b) \in R \text{ and } (b,c) \in R] \rightarrow (a,c) \in R$ for all $a, b, c \in A$.

- **Example** 3:
- Relation R_{fun} on A = {1,2,3,4} defined as:
  - R_{fun} = {(1,2),(2,2),(3,3)}.
- **Is R_{fun} transitive?**
- **Answer: Yes.** It is transitive.

Combining relations

Definition: Let A and B be sets. A **binary relation from A to B** is a subset of a Cartesian product A x B.

- Let $R \subseteq A \times B$ means R is a set of ordered pairs of the form (a,b) where $a \in A$ and $b \in B$.

Combining Relations

- **Relations are sets → combinations via set operations**
- Set operations of: **union, intersection, difference and symmetric difference.**

Combining relations

Example:

- Let A = {1,2,3} and B = {u,v} and
- R1 = {(1,u), (2,u), (2,v), (3,u)}
- R2 = {(1,v),(3,u),(3,v)}

What is:

- $R1 \cup R2$ = {(1,u),(1,v),(2,u),(2,v),(3,u),(3,v)}
- $R1 \cap R2$ = {(3,u)}
- R1 - R2 = {(1,u),(2,u),(2,v)}
- R2 - R1 = {(1,v),(3,v)}

Combination of relations

- Can the relation formed by taking the union or intersection or composition of two relations R1 and R2 represented in terms of matrix operations? **Yes**

Combination of relations: implementation

Definition. The **join,** denoted by $\vee$, of two m-by-n matrices (a_{ij}) and (b_{ij}) of 0s and 1s is an m-by-n matrix (m_{ij}) where

- $m_{ij} = a_{ij} \vee b_{ij}$ for all i,j

= pairwise or (disjunction)

- **Example**:
- Let A = {1,2,3} and B = {u,v} and
- R1 = {(1,u), (2,u), (2,v), (3,u)}
- R2 = {(1,v),(3,u),(3,v)}

- **MR1** = $\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}$ **MR2** = $\begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$ **M(R1 $\vee$ R2)** = $\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$

Combination of relations: implementation

Definition. The **meet,** denoted by $\wedge$, of two m-by-n matrices (a_{ij}) and (b_{ij}) of 0s and 1s is an m-by-n matrix (m_{ij}) where

- $m_{ij} = a_{ij} \wedge b_{ij}$ for all i,j
  = **pairwise and (conjunction)**

- **Example**:
- Let A = {1,2,3} and B = {u,v} and
- R1 = {(1,u), (2,u), (2,v), (3,u)}
- R2 = {(1,v),(3,u),(3,v)}

- **MR1** $=\begin{bmatrix}1 & 0\\1 & 1\\1 & 0\end{bmatrix}$ **MR2** $=\begin{bmatrix}0 & 1\\0 & 0\\1 & 1\end{bmatrix}$ **MR1** $\wedge$ **MR2** $=\begin{bmatrix}0 & 0\\0 & 0\\1 & 0\end{bmatrix}$