CS 441 Discrete Mathematics for CS
Lecture 20

Probabilities II

Milos Hauskrecht
milos@cs.pitt.edu
5329 Sennott Square

Random variables

- **Definition: A random variable** is a function from the sample space of an experiment to the set of real numbers f: S → R. A random variable assigns a number to each possible outcome.
- **The distribution of a random variable X on the sample space S** is a set of pairs (r p(X=r)) for all r in S where r is the number and p(X=r) is the probability that X takes a value r.

Random variables

Example:

Let S be the outcomes of a two-dice roll

Let random variable X denotes the sum of outcomes

(1,1) → 2

(1,2) and (2,1) → 3

(1,3), (3,1) and (2,2) → 4

…

Distribution of X:

- 2 → 1/36,
- 3 → 2/36,
- 4 → 3/36 …
- 12 → 1/36

Probabilities

- **Assume a repeated coin flip**
- P(head) =0.6 and the probability of a tail is 0.4. Each coin flip is independent of the previous.
- What is the probability of seeing:
  - HHHHH - 5 heads in a row
- **P(HHHHH) = 0.6^5 =**
  - Assume the outcome is HHTTT
- **P(HHTTT)= $0.6*0.6*0.4^{3}= 0.6^2*0.4^3$**
  - Assume the outcome is TTHHT
- **P(TTHHT)=$0.4^{2}*0.6^2*0.4 = 0.6^2*0.4^3$**
- What is the probability of seeing three tails and two heads?
- The number of two-head-three tail combinations = **C(5,2)**
- P(two-heads-three tails) = **C(5,2) $*0.6^2*0.4^3$**

Probabilities

- **Assume a variant of a repeated coin flip problem**
- The space of possible outcomes is the count of occurrences of heads in 5 coin flips. For example:
- TTTTT yields outcome 0
- HTTTT or TTHTT yields 1
- HTHHT yields 3 …

- What is the probability of an outcome 0?
- P(outcome=0) = $\mathbf{0.6^0 * 0.4^5}$
- P(outcome=1) = $\mathbf{C(5,1)\ 0.6^1 * 0.4^4}$
- P(outcome =2) = $\mathbf{C(5,2)\ 0.6^2 * 0.4^3}$
- P(outcome =3) = $\mathbf{C(5,3)\ 0.6^3 * 0.4^2}$
- …

Expected value and variance

Definition: The expected value of the random variable X(s) on the sample space is equal to:

$$E(X) = \sum_{s \in S} p(s) X(s)$$

Example: roll of a dice

- Outcomes: 1 2 3 4 5 6
- Expected value:

  E(X) = 1*1/6 + 2*1/6+3*1/6 + 4*1/6 + 5*1/6 +6* 1/6 = 7/2=3.5

Expected value

Example:

Flip a fair coin 3 times. The outcome of the trial X is the number of heads. What is the expected value of the trial?

Answer:

Possible outcomes:

= {HHH HHT HTH THH HTT THT TTH TTT}

3	**2**	**2**	**2**	**1**	**1**	**1**	**0**

E(X) = 1/8 (3 + 3*2 +3*1 +0) = 12/8= 3/2

Expected value

- **Theorem:** If Xi i=1,2,3, n with n being a positive integer, define random variables on S, and a and b are real numbers then:
  - E(X1+X2+ ... Xn) = E(X1)+E(X2) + ... E(Xn)
  - E(aX+b) = aE(X) +b

Expected value

Example:

- Roll a pair of dices. What is the expected value of the sum of outcomes?
- **Approach 1:**
- Outcomes: (1,1) (1,2) (1,3) .... (6,1)... (6,6)

  2 3 4 7 12

Expected value: 1/36 (2*1 + ...) = 7

- **Approach 2 (theorem):**
- $E(X1+X2) = E(X1) + E(X2)$
- $E(X1) = 7/2$ $E(X2) = 7/2$
- $E(X1+X2) = 7$

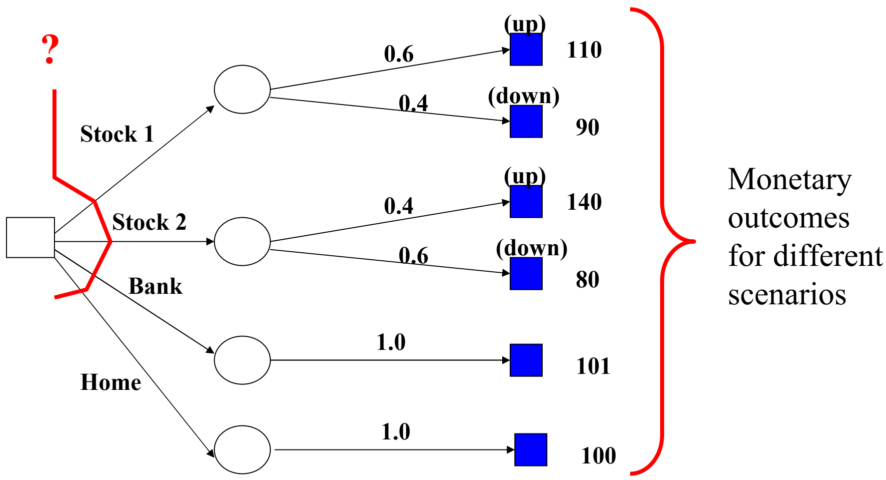
Expected value

Investment problem:

- You have 100 dollar and can invest into a stock. The returns are volatile and you may get either $120 with probability of 0.4, or $90 with probability 0.6.
- **What is the expected value of your investment?**
- $E(X) = 0.4*120+0.6*90=48+54=102$
- **Is it OK to invest?**

Decision making example

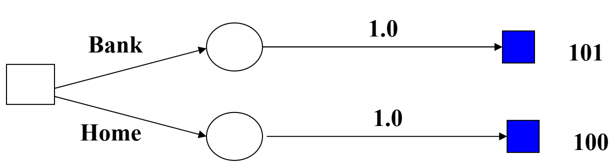
We need to make a choice whether to invest in Stock 1 or 2, put money into bank or keep them at home.



Decision making example.

Assume the simplified problem with the Bank and Home choices only.

The result is guaranteed – the outcome is deterministic

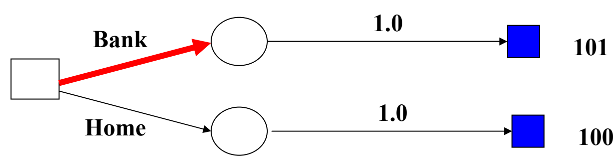


What is the rational choice assuming our goal is to make money?

Decision making. Deterministic outcome.

Assume the simplified problem with the Bank and Home choices only.

These choices are deterministic.

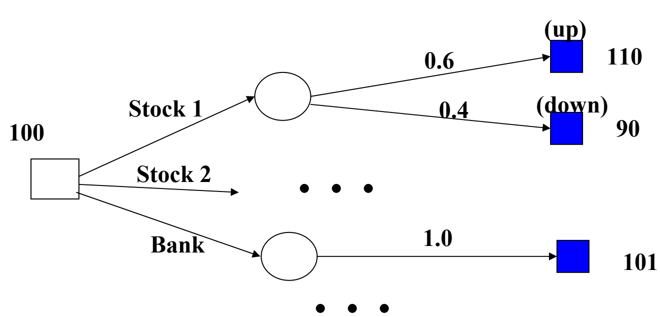


Our goal is to make money. What is the rational choice?

Answer: Put money into the bank. The choice is always strictly better in terms of the outcome

Decision making

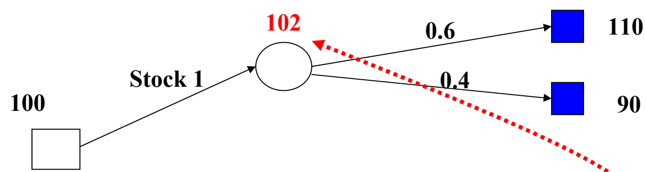
- **How to quantify the goodness of the stochastic outcome?**
  We want to compare it to deterministic and other stochastic outcomes.



Idea: Use the expected value of the outcome

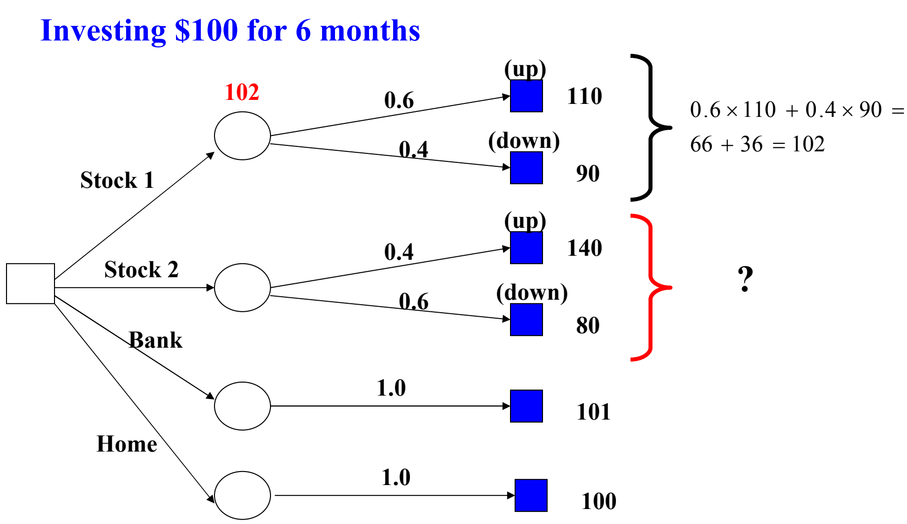
Expected value

- **Expected value** summarizes all stochastic outcomes into a single quantity
  - Expected value for the outcome of the Stock 1 option is:



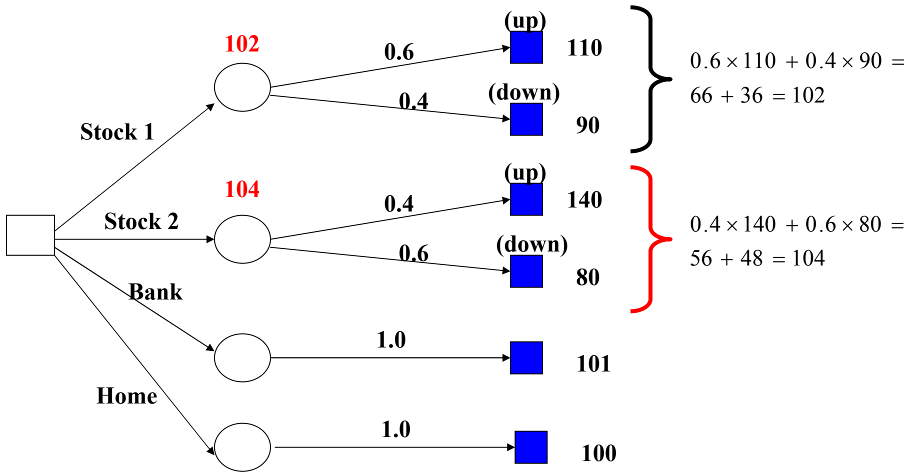
$$0.6 \times 110 + 0.4 \times 90 = 66 + 36 = \boxed{102}$$

Expected values



Expected values

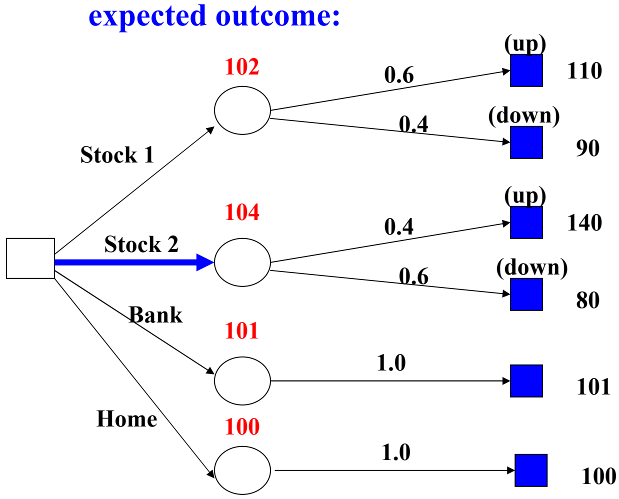
Investing $100 for 6 months



M. Hauskrecht

Selection based on expected values

The optimal action is the option that maximizes the expected outcome:



M. Hauskrecht