CS 441 Discrete Mathematics for CS
Lecture 2

Propositional logic

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Course administration

Homework 1

- Is out and posted on the course web page
- Due on Thursday, September 10, 2009.

Recitations this week:

- Friday, September 4, 2009
- 5313 Sennott Square Bldg.

Course web page:

http://www.cs.pitt.edu/~milos/courses/cs441/

Propositional logic: review

- **Propositional logic:** a formal language for representing knowledge and for making logical inferences
- **A proposition is** a statement that is either true or false.
- **A compound proposition** can be created from other propositions using logical connectives
- **The truth of a compound proposition** is defined by truth values of elementary propositions and the meaning of connectives.
- **The truth table for a compound proposition:** table with entries (rows) for all possible combinations of truth values of elementary propositions.

Compound propositions

- More complex propositional statements can be build from the elementary statements using **logical connectives.**

- **Logical connectives:**
  - Negation
  - Conjunction
  - Disjunction
  - Exclusive or
  - Implication
  - Biconditional

Compound propositions

- Let p: 2 is a prime ….. **T**
  q: 6 is a prime ….. **F**
- Determine **the truth value** of the following statements:

  $\neg p$: **F**

  $p \wedge q$: **F**

  $p \wedge \neg q$: **T**

  $p \vee q$: **T**

  $p \oplus q$: **T**

  $p \rightarrow q$: **F**

  $q \rightarrow p$: **T**

Constructing the truth table

- **Example:** Construct the truth table for
  $(p \rightarrow q) \wedge (\neg p \leftrightarrow q)$

p	q	¬p	p → q	¬p ↔ q	(p→q)∧ (¬p↔q)
T	T				
T	F				
F	T				
F	F				

Constructing the truth table

- Example: Construct the truth table for $(p \rightarrow q) \wedge (\neg p \leftrightarrow q)$

p	q	¬p			(p→q)∧(¬p↔q)
T	T				
T	F				
F	T				
F	F				

Rows: all possible combinations of values for elementary propositions: 2^n values

Constructing the truth table

- Example: Construct the truth table for $(p \rightarrow q) \wedge (\neg p \leftrightarrow q)$

Typically the target (unknown) compound proposition and its values

p	q	¬p	p → q	¬p ↔ q	(p→q)∧(¬p↔q)
T	T				
T	F				
F	T				
F	F				

Auxiliary compound propositions and their values

Constructing the truth table

- Examples: Construct a truth table for
  $(p \rightarrow q) \wedge (\neg p \leftrightarrow q)$

p	q	$\neg p$	$p \rightarrow q$	$\neg p \leftrightarrow q$	$(p\rightarrow q)\wedge(\neg p\leftrightarrow q)$
T	T	F	T	F	F
T	F	F	F	T	F
F	T	T	T	T	T
F	F	T	T	F	F

Translation

Logic helps us to define the meaning of statements:
- Mathematical or English statements.

Question: **How to translate an English sentence to the logic?**

Assume a sentence:
- If you are older than 13 or you are with your parents then you can attend a PG-13 movie.

- The whole sentence is a proposition. It is **True.**

- But this is not the best. We want to parse the sentence to elementary statements that are combined with connectives.

Translation

If you are older than 13 or you are with your parents then you can attend a PG-13 movie.

Parse:

- **If** **(** you are older than 13 **or** you are with your parents **)** **then** **(** you can attend a PG-13 movie**)**
  - A= you are older than 13
  - B= you are with your parents
  - C=you can attend a PG-13 movie
- **Translation:** $\mathbf{A \vee B \rightarrow C}$
- Why do we want to do this?
- **Inference:** Assume I know that $A \vee B \rightarrow C$ is true and A is true. Then we can conclude that C is true as well.
- $A \vee B \rightarrow C$ and A are both true then C is true

Translation

- **General rule for translation.**
- Look for patterns corresponding to logical connectives in the sentence and use them to define elementary propositions.

- **Example:**

You can have free coffee if you are senior citizen and it is a Tuesday

Step 1 find logical connectives

Translation

- **General rule for translation.**
- Look for patterns corresponding to logical connectives in the sentence and use them to define elementary propositions.

- **Example:**

You can have free coffee if you are senior citizen and it is a Tuesday

a **b** **c**

Step 2 break the sentence into elementary propositions

Translation

- **General rule for translation .**
- Look for patterns corresponding to logical connectives in the sentence and use them to define elementary propositions.

- **Example:**

You can have free coffee if you are senior citizen and it is a Tuesday

a **b** **c**

Step 3 rewrite the sentence in propositional logic

$$\mathbf{b} \wedge \mathbf{c} \rightarrow \mathbf{a}$$

Translation

- Assume two elementary statements:
  - **p: you drive over 65 mph** ; **q: you get a speeding ticket**
- **Translate each of these sentences to logic**
  - you do not drive over 65 mph. $(\neg p)$
  - you drive over 65 mph, but you don't get a speeding ticket. $(p \wedge \neg q)$
  - you will get a speeding ticket if you drive over 65 mph. $(p \rightarrow q)$
  - if you do not drive over 65 mph then you will not get a speeding ticket. $(\neg p \rightarrow \neg q)$
  - driving over 65 mph is sufficient for getting a speeding ticket. $(p \rightarrow q)$
  - you get a speeding ticket, but you do not drive over 65 mph. $\mathbf{(q \wedge \neg p)}$

Computer representation of True and False

We need to encode two values True and False:

- **Computers represents data and programs using 0s and 1s**
- Logical truth values – True and False
- A bit is sufficient to represent two possible values:
  - 0 (False) or 1(True)

- A variable that takes on values 0 or 1 is called a **Boolean variable**.

- Definition: A **bit string** is a sequence of zero or more bits. The **length** of this string is the number of bits in the string.

Bitwise operations

- T and F replaced with 1 and 0

p	q	$p \vee q$	$p \wedge q$
1	1	1	1
1	0	1	0
0	1	1	0
0	0	0	0

p	$\neg p$
1	0
0	1

Bitwise operations

- **Examples:**

$$\begin{array}{r} 1011\ 0011 \\ \vee\ \underline{0110\ 1010} \end{array} \qquad \begin{array}{r} 1011\ 0011 \\ \wedge\ \underline{0110\ 1010} \end{array} \qquad \begin{array}{r} 1011\ 0011 \\ \oplus\ \underline{0110\ 1010} \end{array}$$

Bitwise operations

- **Examples:**

$$\begin{array}{r} 1011\ 0011 \\ \vee\ \underline{0110\ 1010} \\ 1111\ 1011 \end{array} \qquad \begin{array}{r} 1011\ 0011 \\ \wedge\ \underline{0110\ 1010} \\ \end{array} \qquad \begin{array}{r} 1011\ 0011 \\ \oplus\ \underline{0110\ 1010} \\ \end{array}$$

Bitwise operations

- **Examples:**

$$\begin{array}{r} 1011\ 0011 \\ \vee\ \underline{0110\ 1010} \\ 1111\ 1011 \end{array} \qquad \begin{array}{r} 1011\ 0011 \\ \wedge\ \underline{0110\ 1010} \\ 0010\ 0010 \end{array} \qquad \begin{array}{r} 1011\ 0011 \\ \oplus\ \underline{0110\ 1010} \\ 1101\ 1001 \end{array}$$

Tautology and Contradiction

- Some propositions are interesting since their values in the truth table are always the same

Definitions:

- A compound proposition that is always true for all possible truth values of the propositions is called a **tautology**.
- A compound proposition that is always false is called a **contradiction**.
- A proposition that is neither a tautology nor contradiction is called a **contingency**.

Example: $p \vee \neg p$ is a **tautology.**

p	$\neg p$	$p \vee \neg p$
T	F	T
F	T	T

Tautology and Contradiction

- Some propositions are interesting since their values in the truth table are always the same

Definitions:

- A compound proposition that is always true for all possible truth values of the propositions is called a **tautology**.
- A compound proposition that is always false is called a **contradiction**.
- A proposition that is neither a tautology nor contradiction is called a **contingency**.

Example: $p \wedge \neg p$ is a **contradiction.**

p	$\neg p$	$p \wedge \neg p$
T	F	F
F	T	F

Equivalence

- We have seen that some of the propositions are equivalent. Their truth values in the truth table are the same.
- Example: $\mathbf{p \rightarrow q}$ is equivalent to $\mathbf{\neg q \rightarrow \neg p}$ **(contrapositive)**

p	q	$p \rightarrow q$	$\neg q \rightarrow \neg p$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

- Equivalent statements are important for logical reasoning since they can be substituted and can help us to make a logical argument.

Logical equivalence

Definition: The propositions p and q are called **logically equivalent** if $p \leftrightarrow q$ is a tautology (alternately, if they have the same truth table). The notation **p <=> q** denotes p and q are logically equivalent.

Example of important equivalences

- **DeMorgan's Laws:**
- 1) $\neg(p \vee q) \iff \neg p \wedge \neg q$
- 2) $\neg(p \wedge q) \iff \neg p \vee \neg q$

Example: Negate "The summer in Mexico is cold and sunny" with DeMorgan's Laws

Solution: ?

Equivalence

- **Definition**: The propositions p and q are called **logically equivalent** if p ↔ q is a tautology (alternately, if they have the same truth table). The notation **p <=> q** denotes p and q are logically equivalent.

Example of important equivalences

- **DeMorgan's Laws:**
- 1) $\neg(p \vee q) \iff \neg p \wedge \neg q$
- 2) $\neg(p \wedge q) \iff \neg p \vee \neg q$

Example: Negate "The summer in Mexico is cold and sunny" with DeMorgan's Laws

Solution: "The summer in Mexico is not cold or not sunny."

Equivalence

Example of important equivalences

- **DeMorgan's Laws:**
- 1) $\neg(p \vee q) \iff \neg p \wedge \neg q$
- 2) $\neg(p \wedge q) \iff \neg p \vee \neg q$

To convince us that two propositions are logically equivalent use the truth table

p	q	¬p	¬q	¬(p ∨ q)	¬p ∧ ¬q
T	T	F	F		
T	F	F	T		
F	T	T	F		
F	F	T	T		

Equivalence

Example of important equivalences

- **DeMorgan's Laws:**
- 1) $\neg(p \lor q) \iff \neg p \land \neg q$
- 2) $\neg(p \land q) \iff \neg p \lor \neg q$

To convince us that two propositions are logically equivalent use the truth table

p	q	¬p	¬q	¬(p ∨ q)	¬p ∧ ¬q
T	T	F	F	F	
T	F	F	T	F	
F	T	T	F	F	
F	F	T	T	T	

Equivalence

Example of important equivalences

- **DeMorgan's Laws:**
- 1) $\neg(p \lor q) \iff \neg p \land \neg q$
- 2) $\neg(p \land q) \iff \neg p \lor \neg q$

To convince us that two propositions are logically equivalent use the truth table

p	q	¬p	¬q	¬(p ∨ q)	¬p ∧ ¬q
T	T	F	F	F	F
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

Equivalence

Example of important equivalences

- **DeMorgan's Laws:**
- 1) $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$
- 2) $\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$

To convince us that two propositions are logically equivalent use the truth table

p	q	$\neg p$	$\neg q$	$\neg(p \vee q)$	$\neg p \wedge \neg q$
T	T	F	F	F	F
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

Important logical equivalences

- **Identity**
  - $p \wedge T \Leftrightarrow p$
  - $p \vee F \Leftrightarrow p$

- **Domination**
  - $p \vee T \Leftrightarrow T$
  - $p \wedge F \Leftrightarrow F$

- **Idempotent**
  - $p \vee p \Leftrightarrow p$
  - $p \wedge p \Leftrightarrow p$

Important logical equivalences

- **Double negation**
  - $\neg(\neg p) \iff p$
- **Commutative**
  - $p \vee q \iff q \vee p$
  - $p \wedge q \iff q \wedge p$
- **Associative**
  - $(p \vee q) \vee r \iff p \vee (q \vee r)$
  - $(p \wedge q) \wedge r \iff p \wedge (q \wedge r)$

Important logical equivalences

- **Distributive**
  - $p \vee (q \wedge r) \iff (p \vee q) \wedge (p \vee r)$
  - $p \wedge (q \vee r) \iff (p \wedge q) \vee (p \wedge r)$
- **De Morgan**
  - $\neg(p \vee q) \iff \neg p \wedge \neg q$
  - $(p \wedge q) \iff \neg p \vee \neg q$
- **Other useful equivalences**
  - $p \vee \neg p \iff T$
  - $p \wedge \neg p \iff F$
  - **$p \rightarrow q \iff (\neg p \vee q)$**