CS 441 Discrete Mathematics for CS
Lecture 19

Midterm exam 2 review

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Course administration

- **Midterm exam 2**
  - **Thursday, November 12, 2009**
  - **Covers only the material after midterm 1**
    - **Sequences and Summations**
    - **Integers (Primes, Division, Congruencies)**
    - **Inductive proofs and Recursion**
    - **Counting**

Course web page:
http://www.cs.pitt.edu/~milos/courses/cs441/

Review

- **Sequences and Summations**
  - Arithmetic and Geometric progression
  - Summations. Arithmetic and Geometric series.
- **Integers**
  - Primes
  - Division, greatest common divisor, least common multiple
  - Congruencies and applications
  - Countable sets
- **Inductive proofs and recursion**
- **Counting**
  - Basic rules
  - Pigeonhole principle
  - Permutations and Combinations
  - Binomial coefficients

Review questions

Sequences

Question:

$a_n = n^2$, where $n = 1,2,3...$

What are the elements of the sequence.

Review questions

Sequences

Question:

$a_n = n^2$, where $n = 1,2,3...$

What are the elements of the sequence?

1, 4, 9, 16, 25, ...

Review questions

Sequences

Question:

How is an arithmetic progression defined?

Review questions

Sequences

Question:

How is the arithmetic progression defined?

$a_n = a + nd$ for $n=0,1,2\ldots$

a, a+d,a+2d, …, a+nd

where a is the *initial term* and d is *common difference*, such that both belong to R.

Example:

- $s_n = -1+4n$

Review questions

Sequences

Question:

How is the arithmetic progression defined?

$a_n = a + nd$ for $n=0,1,2\ldots$

a, a+d,a+2d, …, a+nd

where a is the *initial term* and d is *common difference*, such that both belong to R.

Example:

- $s_n = -1+4n$

Sequence: -1, 3, 7, 11, 15, …

Review questions

Sequences

Question:

Assume a sequence:

2 5 8 11 14 …

What type of a sequence is this?

Review questions

Sequences

Question:

Assume a sequence:

2 5 8 11 14 …

What type of a sequence is this?

- Aritmetic progression:

  $a_n = a + nd$ for $n=0,1,2\ldots$

  $a_n = 2 + 3n$ for $n=0,1,2\ldots$

Review question

Sequences;

How is a geometric progression defined?

Review questions

Sequences;

How is a geometric progression defined?

A **geometric progression** is a sequence of the form:

$$a, ar, ar^2, \ldots, ar^k,$$

where a is the *initial term*, and r is the *common ratio*. Both a and r belong to R.

Example:

- $a_n = (\frac{1}{2})^n$

Review questions

Sequences;

How is a geometric progression defined?

A **geometric progression** is a sequence of the form:

a, ar, ar^2, ..., ar^k,

where a is the *initial term*, and r is the *common ratio*. Both a and r belong to R.

Example:

- $a_n = (\frac{1}{2})^n$
  members: 1,½, ¼, 1/8, …..

Review questions

Summations

Formula for arithmetic series?

$$S = \sum_{j=1}^{n} j = \quad ?$$

Review questions

Summations

Formula for arithmetic series?

$$S = \sum_{j=1}^{n} j = \frac{n(n+1)}{2}$$

Review questions

Summations

Formula for arithmetic series?

$$S = \sum_{j=1}^{n} j = \frac{n(n+1)}{2}$$

$$S = \sum_{j=1}^{n} (a + jd) = na + d\sum_{j=1}^{n} j = na + d\,\frac{n(n+1)}{2}$$

Arithmetic series

Example: $S = \sum_{j=1}^{5} (2 + j3) =$

Arithmetic series

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$$= \sum_{j=1}^{5} 2 + \sum_{j=1}^{5} j3 =$$

Arithmetic series

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$$S = \sum_{j=1}^{5}(2 + j3) =$$
$$= \sum_{j=1}^{5} 2 + \sum_{j=1}^{5} j3 =$$
$$= 2\sum_{j=1}^{5} 1 + 3\sum_{j=1}^{5} j =$$

Arithmetic series

Example:

$$S = \sum_{j=1}^{5}(2 + j3) =$$
$$= \sum_{j=1}^{5} 2 + \sum_{j=1}^{5} j3 =$$
$$= 2\sum_{j=1}^{5} 1 + 3\sum_{j=1}^{5} j =$$
$$= 2*5 + 3\sum_{j=1}^{5} j =$$

Arithmetic series

Example:

$$S = \sum_{j=1}^{5}(2 + j3) =$$
$$= \sum_{j=1}^{5} 2 + \sum_{j=1}^{5} j3 =$$
$$= 2\sum_{j=1}^{5} 1 + 3\sum_{j=1}^{5} j =$$
$$= 2*5 + 3\sum_{j=1}^{5} j =$$
$$= 10 + 3\frac{(5+1)}{2}*5 =$$

Arithmetic series

Example:

$$S = \sum_{j=1}^{5}(2 + j3) =$$
$$= \sum_{j=1}^{5} 2 + \sum_{j=1}^{5} j3 =$$
$$= 2\sum_{j=1}^{5} 1 + 3\sum_{j=1}^{5} j =$$
$$= 2*5 + 3\sum_{j=1}^{5} j =$$
$$= 10 + 3\frac{(5+1)}{2}*5 =$$
$$= 10 + 45 = 55$$

Arithmetic series

Example 2:

$$S = \sum_{j=3}^{5}(2 + j3) =$$

Arithmetic series

Example 2:

$$S = \sum_{j=3}^{5}(2 + j3) =$$
$$= \left[\sum_{j=1}^{5}(2 + j3)\right] - \left[\sum_{j=1}^{2}(2 + j3)\right]$$

Trick

Arithmetic series

Example 2: $S = \sum_{j=3}^{5}(2 + j3) =$

$$= \left[\sum_{j=1}^{5}(2 + j3)\right] - \left[\sum_{j=1}^{2}(2 + j3)\right]$$ ⇐ **Trick**

$$= \left[2\sum_{j=1}^{5}1 + 3\sum_{j=1}^{5} j\right] - \left[2\sum_{j=1}^{2}1 + 3\sum_{j=1}^{2} j\right]$$

$$= 55 - 13 = 42$$

Double summations

Example: $S = \sum_{i=1}^{4}\sum_{j=1}^{2}(2i - j) =$

Double summations

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Double summations

Example: $S = \sum_{i=1}^{4}\sum_{j=1}^{2}(2i - j) =$

$$= \sum_{i=1}^{4}\left[\sum_{j=1}^{2}2i - \sum_{j=1}^{2} j\right] =$$

$$= \sum_{i=1}^{4}\left[2i\sum_{j=1}^{2}1 - \sum_{j=1}^{2} j\right] =$$

Double summations

Example: $S = \sum_{i=1}^{4}\sum_{j=1}^{2}(2i - j) =$

$$= \sum_{i=1}^{4}\left[\sum_{j=1}^{2}2i - \sum_{j=1}^{2} j\right] =$$

$$= \sum_{i=1}^{4}\left[2i\sum_{j=1}^{2}1 - \sum_{j=1}^{2} j\right] =$$

$$= \sum_{i=1}^{4}\left[2i * 2 - \sum_{j=1}^{2} j\right] =$$

$$= \sum_{i=1}^{4}[2i * 2 - 3] =$$

Double summations

Example: $S = \sum_{i=1}^{4}\sum_{j=1}^{2}(2i - j) =$

$$= \sum_{i=1}^{4}\left[\sum_{j=1}^{2}2i - \sum_{j=1}^{2} j\right] =$$

$$= \sum_{i=1}^{4}\left[2i\sum_{j=1}^{2}1 - \sum_{j=1}^{2} j\right] =$$

$$= \sum_{i=1}^{4}\left[2i * 2 - \sum_{j=1}^{2} j\right] =$$

$$= \sum_{i=1}^{4}[2i * 2 - 3] =$$

$$= \sum_{i=1}^{4}4i - \sum_{i=1}^{4}3 =$$

Double summations

Example: $S=\sum_{i=1}^{4}\sum_{j=1}^{2}(2i-j)=$

$$=\sum_{i=1}^{4}\left[\sum_{j=1}^{2}2i-\sum_{j=1}^{2}j\right]=$$

$$=\sum_{i=1}^{4}\left[2i\sum_{j=1}^{2}1-\sum_{j=1}^{2}j\right]=$$

$$=\sum_{i=1}^{4}\left[2i*2-\sum_{j=1}^{2}j\right]=$$

$$=\sum_{i=1}^{4}[2i*2-3]=$$

$$=\sum_{i=1}^{4}4i-\sum_{i=1}^{4}3=$$

$$=4\sum_{i=1}^{4}i-3\sum_{i=1}^{4}1=$$

Double summations

Example: $S=\sum_{i=1}^{4}\sum_{j=1}^{2}(2i-j)=$

$$=\sum_{i=1}^{4}\left[\sum_{j=1}^{2}2i-\sum_{j=1}^{2}j\right]=$$

$$=\sum_{i=1}^{4}\left[2i\sum_{j=1}^{2}1-\sum_{j=1}^{2}j\right]=$$

$$=\sum_{i=1}^{4}\left[2i*2-\sum_{j=1}^{2}j\right]=$$

$$=\sum_{i=1}^{4}[2i*2-3]=$$

$$=\sum_{i=1}^{4}4i-\sum_{i=1}^{4}3=$$

$$=4\sum_{i=1}^{4}i-3\sum_{i=1}^{4}1= \quad 4*10-3*4=28$$

Review questions

Summations

Formula for geometric series?

Review questions

Summations

Formula for the geometric series?

$$S=\sum_{j=0}^{n}(ar^{j})=a\sum_{j=0}^{n}r^{j}= \ \mathbf{?}$$

Review questions

Summations

Formula for geometric series?

$$S=\sum_{j=0}^{n}(ar^{j})=a\sum_{j=0}^{n}r^{j}=a\left[\frac{r^{n+1}-1}{r-1}\right]$$

Review questions

Summations

Formula for an infinite geometric series?

Review questions

Summations

Formula for infinite geometric series?

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

If 0 < x < 1

Review questions

Fundamental theorem of Arithmetic:

Review questions

Fundamental theorem of Arithmetic:

- Any positive integer greater than 1 can be expressed as a product of prime numbers.

Review questions

Is a number a prime?

Question: is 97 a prime?

Review questions

Is a number a prime?

Question: is 97 a prime?

Approach 1: try all positive integers < 97

Review questions

Is a number a prime?

Question: is 97 a prime?

Approach 1: try all positive integers < 97

Approach 2: try all primes < 97

Review questions

Is a number a prime?

Question: is 97 a prime?

Approach 1: try all positive integers < 97

Approach 2: try all primes < 97

Approach 3: try all primes smaller than $\sqrt{97}$

Review questions

Finding the greatest common divisor of two numbers

Question: what is the gcd of 233 and 541

Review questions

Finding the greatest common divisor of two numbers

Question: what is the gcd of 233 and 541

Approach 1: factorization and minimum of powers

Approach 2: Euclid algorithm

Review questions

Congruencies

Question: is 3 and 7 congruent modulo 4?

Review questions

Congruencies

Question: is 3 and 7 congruent modulo 4?

3 mod 4=3

7 mod 4=3

Yes they are congruent.

Review questions

Countable sets

Is a finite set countable?

Review questions

Countable sets

Is a finite set countable?

Yes.

What other sets are called countable?

Review questions

Countable sets

Is a finite set countable?

Yes.

What other sets are called countable?

A set that has the same cardinality as the set of positive integers Z^+

Review questions

Countable sets

Is a finite set countable?

Yes.

What other (infinite) sets are called countable?

A set that has the same cardinality as the set of positive integers Z^+

How to show the countability of infinite sets?

Review questions

Countable sets

Is a finite set countable?

Yes.

What other (infinite) sets are called countable?

A set that has the same cardinality as the set of positive integers Z^+

How to show the countability of infinite sets?

Show that a bijection in between Z^+ and the set exists

Review questions

Example:

- Assume A = {0, 2, 4, 6, ... } set of even numbers. Is it countable?

Review questions

Example:

- Assume A = {0, 2, 4, 6, ... } set of even numbers. Is it countable?
- Using the definition: Is there a bijective function f: $Z^+ \rightarrow A$

Z+ = {1, 2, 3, 4, …}

Review questions

Example:

- Assume A = {0, 2, 4, 6, ... } set of even numbers. Is it countable?
- Using the definition: Is there a bijective function f: $Z^+ \rightarrow A$

Z+ = {1, 2, 3, 4, …}

- Define a function f: $x \rightarrow 2x - 2$ (an arithmetic progression)
  - $1 \rightarrow 2(1)-2 = 0$
  - $2 \rightarrow 2(2)-2 = 2$
  - $3 \rightarrow 2(3)-2 = 4$ ...

Review questions

Example:

- Assume A = {0, 2, 4, 6, ... } set of even numbers. Is it countable?
- Using the definition: Is there a bijective function f: $Z^+ \rightarrow A$

Z+ = {1, 2, 3, 4, …}

- Define a function f: $x \rightarrow 2x - 2$ (an arithmetic progression)
  - $1 \rightarrow 2(1)-2 = 0$
  - $2 \rightarrow 2(2)-2 = 2$
  - $3 \rightarrow 2(3)-2 = 4$ ...
- one-to-one (why?) 2x-2 = 2y-2 => 2x = 2y =>x = y.
- onto (why?) $\forall a \in A$, (a+2) / 2 is the pre-image in Z^+.
- Therefore | A | = | Z^+ |.

Review questions

Countable sets

Is the set of real numbers countable?

Review questions

Countable sets

Is the set of real numbers countable?
No !!!

Review questions

Mathematical induction

- Used to prove statements of the form $\forall x\ P(x)$ where $x \in Z^+$
- **What are the two steps of the proof?**

Review questions

Mathematical induction

- Used to prove statements of the form $\forall n\ P(n)$ where $n \in Z^+$
- **What are the two steps of the proof?**

  1) **Basis step:** The proposition P(1) is true.
  2) **Inductive Step:** The implication $P(k) \rightarrow P(k+1)$, is true for all positive k.
- Therefore we conclude $\forall n\ P(n)$.

Mathematical induction

Example: Prove the sum of first n odd integers is n^2.

i.e. $1 + 3 + 5 + 7 + ... + (2n - 1) = n^2$ for all positive integers.

Proof:

- What is P(n)? P(n): $1 + 3 + 5 + 7 + ... + (2n - 1) = n^2$

Mathematical induction

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Basic Step

Mathematical induction

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Proof:

- What is P(n)? P(n): $1 + 3 + 5 + 7 + ... + (2n - 1) = n^2$

Basis Step Show P(1) is true

- Trivial: $1 = 1^2$

Mathematical induction

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Inductive Step Show if P(n) is true then P(n+1) is true for all n.

-

Mathematical induction

Example: Prove the sum of first n odd integers is n^2.

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Proof:

- What is P(n)? P(n): $1 + 3 + 5 + 7 + ... + (2n - 1) = n^2$

Basis Step Show P(1) is true

- Trivial: $1 = 1^2$

Inductive Step Show if P(n) is true then P(n+1) is true for all n.

- Suppose P(n) be true, that is $1 + 3 + 5 + 7 + ... + (2n - 1) = n^2$
- Show P(n+1): $1 + 3 + 5 + 7 + ... + (2n - 1) + (2n + 1) = (n+1)^2$ follows:
- $\underbrace{1 + 3 + 5 + 7 + ... + (2n - 1)}_{n^2} + (2n + 1) =$

  $n^2 + (2n+1) = (n+1)^2$

Review questions

Mathematical induction

What is the difference in between regular and strong induction?

Review questions

Mathematical induction

What is the difference in between regular and strong induction?

- **The regular induction:**
  - uses the basic step **P(1)** and
  - inductive step **P(k) → P(k+1)**
- **Strong induction uses:**
  - Uses the basis step **P(1)** and
  - inductive step **P(1) and P(2) … P(k) → P(k+1)**

Review

Recursive function:

a function on the set of nonnegative integers can be defined by

- 1. Specifying the value of the function at 0
- 2. Giving a rule for finding the function's value at n+1 in terms of the function's value at integers $i \leq n$.

Review

- **Example**

  Define the function:

  $f(n) = 2n + 1 \quad n = 0, 1, 2, ...$

  recursively.

- $f(0) = ?$

Review

- **Example**

  Define the function:

  $f(n) = 2n + 1 \quad n = 0, 1, 2, ...$

  recursively.

- $f(0) = 1$
- $f(n+1) = ?$

Review

- **Example:**

  Define the function:

  $f(n) = 2n + 1 \quad n = 0, 1, 2, ...$

  recursively.

- $f(0) = 1$
- $f(n+1) = f(n) + 2$

Review

Counting

Basic counting rules?

Review questions

Counting

Basic counting rules?
- Product rule
- Sum rule

How do we count with the product rule?

Review questions

Counting

Basic counting rules?
- Product rule
- Sum rule

How do we count with product rule?
  - *n = n1*n2* ...* nk*

k dependent counts

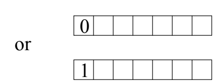
Review questions

Counting

Example:
- How many different bit strings of length 7 are there?
  - E.g. 1011010
- Is it possible to decompose the count problem and if yes how?

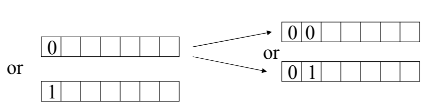
Review

Example:
- How many different bit strings of length 7 are there?
  - E.g. 1011010
- Is it possible to decompose the count problem and if yes how?
- **Yes.**
  - Count the number of possible assignments to bit 1



Review

Example:
- How many different bit strings of length 7 are there?
  - E.g. 1011010
- Is it possible to decompose the count problem and if yes how?
- **Yes.**
  - Then count the number of possible assignments to bit 2



Total assignments to first 2 bits: 2*2=4

Review

Example:
- How many different bit strings of length 7 are there?
  - E.g. 1011010
- Is it possible to decompose the count problem and if yes how?
- **Yes.**
  - Then count the number of possible assignments to bit 7

$n = 2 * 2 * 2 * 2 * 2 * 2 * 2 = 2^7$

Review questions

Counting

What is the pigeonhole principle?

Review questions

Counting

What is the pigeonhole principle?

- **If there are k+1 objects and k bins. Then there is at least one bin with two or more objects.**

Review questions

Counting

Theorem: If N objects are placed into k bins then there is at least one bin containing at least $\lceil N/k \rceil$ objects.

Example:

There are 400 people. What can you say about their birthdays?

Review questions

Counting

Theorem: If N objects are placed into k bins then there is at least one bin containing at least $\lceil N/k \rceil$ objects.

Example:

There are 400 people. What can you say about their birthdays?

There are at least 2 people who have the same birthday.

Review questions

Counting

What are Permutations?

Review questions

Counting

What are Permutations?

Ordered arrangements of n objects

Example: **Assume S={A, B, C}**

Permutations: ?

Review questions

Counting

What are Permutations?

Ordered arrangements of n objects

Example: **Assume S={A, B, C}**

Permutations:

ABC ACB BAC BCA CAB CBA

How many: ?

Review questions

Counting

What are Permutations?

Ordered arrangements of n objects

Example: **Assume S={A, B, C}**

Permutations:

ABC ACB BAC BCA CAB CBA

How many: n!

3!=3*2=6

Review questions

Counting

What are k-combinations from set S?

Review questions

Counting

What are k-combinations from set S?

- Unordered collections of k elements from the set of S

Review questions

Counting

What are k-combinations from set S?

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Example: S={A, B, C}

2-combinations:

Review questions

Counting

What are k-combinations from set S?

- Unordered collections of k elements from the set of S

Example: S={A, B, C}

2-combinations:

AB AC BC

How many are there?

Review questions

Counting

What are k-combinations from set S?

- Unordered collections of k elements from the set of S

Example: S={A, B, C}

2-combinations:

AB AC BC

How many are there? $C(n,k) = \frac{n!}{(n-k)!\,k!}$

- In our case: 3!/2!=3

Review questions

Counting

Example:

3 goalies, 8 defenders, 12 attackers on the hockey team

How many ways to put together the first line that includes:

One goalie

Two defenders

Three attackers

?