Midterm exam 2 review

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Course administration

- **Midterm exam 2**
  - Thursday, November 12, 2009
  - Covers only the material after midterm 1
  - Sequences and Summations
  - Integers (Primes, Division, Congruencies)
  - Inductive proofs and Recursion
  - Counting

Course web page:
http://www.cs.pitt.edu/~milos/courses/cs441/

Review

- **Sequences and Summations**
  - Arithmetic and Geometric progression
  - Summations. Arithmetic and Geometric series.
- **Integers**
  - Primes
  - Division, greatest common divisor, least common multiple
  - Congruencies and applications
  - Countable sets
- **Inductive proofs and recursion**
- **Counting**
  - Basic rules
  - Pigeonhole principle
  - Permutations and Combinations
  - Binomial coefficients

Review questions

**Sequences**

Question:
\[ a_n = n^2, \quad \text{where} \quad n = 1, 2, 3, \ldots \]
What are the elements of the sequence?
1, 4, 9, 16, 25, ...

**Sequences**

Question:
How is an arithmetic progression defined?
**Sequences**

**Question:** How is the arithmetic progression defined?

\[ a_n = a + nd \quad \text{for} \quad n = 0, 1, 2, \ldots \]

\[ a, a+d, a+2d, \ldots, a+nd \]

where \( a \) is the *initial term* and \( d \) is *common difference*, such that both belong to \( \mathbb{R} \).

**Example:**

• \( s_n = -1 + 4n \)

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**Sequences**

**Question:** Assume a sequence:

2 5 8 11 14 …

What type of a sequence is this?

**Example:**

• Aritmetic progression:

\[ a_n = a + nd \quad \text{for} \quad n = 0, 1, 2, \ldots \]

\[ a_n = 2 + 3n \quad \text{for} \quad n = 0, 1, 2, \ldots \]

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**Sequences**

**Question:** How is a geometric progression defined?

A *geometric progression* is a sequence of the form:

\[ a, ar, ar^2, \ldots, ar^n \]

where \( a \) is the *initial term*, and \( r \) is the *common ratio*. Both \( a \) and \( r \) belong to \( \mathbb{R} \).

**Example:**

• \( a_n = (\frac{1}{2})^n \)
Review questions

Sequences:
How is a geometric progression defined?

A geometric progression is a sequence of the form:
\( a, ar, ar^2, \ldots, ar^n \),
where \( a \) is the initial term, and \( r \) is the common ratio. Both \( a \) and \( r \) belong to \( \mathbb{R} \).

Example:
- \( a_n = (\frac{1}{2})^n \)
- members: \( 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots \)

Review questions

Summations

Formula for arithmetic series?

\[
S = \sum_{j=1}^{n} j = \frac{n(n+1)}{2}
\]

Example:
\[
S = \sum_{j=1}^{5} (2 + j) = 2 + 1 + 2 + 3 + 4 = 14
\]

Review questions

Arithmetic series

Formula for arithmetic series?

\[
S = \sum_{j=1}^{n} j = \frac{n(n+1)}{2}
\]

Example:
\[
S = \sum_{j=1}^{5} (2 + j) = 2 + 1 + 2 + 3 + 4 = 14
\]

\[
S = \sum_{j=1}^{5} (a + jd) = na + d \sum_{j=1}^{n} j = na + d \frac{n(n+1)}{2}
\]
Arithmetic series

Example:
\[ S = \sum_{j=1}^{5} (2 + j) = \]
\[ = \sum_{j=1}^{5} 2 + \sum_{j=1}^{5} j = \]
\[ = \frac{5}{2} \left( 1 + 5 \right) + \frac{5(5+1)}{2} \cdot 5 = \]
\[ = 2 \cdot 5 + 3 \sum_{j=1}^{5} j = \]
\[ = 10 + 3 \cdot \frac{(5+1) \cdot 5}{2} = \]

Trick

Example 2:
\[ S = \sum_{j=1}^{2} (2 + j) = \]
\[ = \left[ \sum_{j=1}^{2} (2 + j) \right] - \left[ \sum_{j=1}^{1} (2 + j) \right] \]
Arithmetic series

Example 2: \[ S = \sum_{j=3}^{5} (2 + j) = \]
\[ = \left[ \sum_{j=3}^{5} (2 + j) \right] - \left[ \sum_{j=3}^{5} j \right] \]
\[ = \left[ 2 \sum_{j=3}^{5} + 3 \sum_{j=3}^{5} j \right] - \left[ 2 \sum_{j=3}^{5} 1 + 3 \sum_{j=3}^{5} j \right] \]
\[ = 55 - 13 = 42 \]

Double summations

Example: \[ S = \sum_{i=1}^{4} \sum_{j=1}^{3} (2i - j) = \]
\[ = \sum_{i=1}^{4} \left[ 2i \sum_{j=1}^{3} - \sum_{j=1}^{3} j \right] = \]
\[ = \sum_{i=1}^{4} \left[ 2i \times 3 - \frac{3 \times (3+1)}{2} \right] = \]
\[ = \sum_{i=1}^{4} \left[ 6i - \frac{6}{2} \right] = \]
\[ = \sum_{i=1}^{4} \left[ 6i - 3 \right] = \]
\[ = 21 \]
Double summations

Example: \[ \sum \sum (2i - j) = \]
\[= \sum i \left( \sum 2 - \sum j \right) = \]
\[= \sum i \left( 2 \sum j - \sum j \right) = \]
\[= \sum i \left( 2 \sum j - \sum j \right) = \]
\[= \sum \left( 2i - j \right) = \]
\[= 4 \sum j - 3 \sum i = \]

Review questions

Summations

Formula for geometric series?

\[ S = \sum_{i=1}^{n} (ar^{i-1}) = \frac{a(r^n - 1)}{r-1} \]
Review questions

Summations

Formula for infinite geometric series?

\[ \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \]

If \( 0 < x < 1 \)

Review questions

Fundamental theorem of Arithmetic:

• Any positive integer greater than 1 can be expressed as a product of prime numbers.

Review questions

Is a number a prime?

Question: is 97 a prime?

Approach 1: try all positive integers < 97

Approach 2: try all primes < 97
Review questions

Is a number a prime?
Question: is 97 a prime?

Approach 1: try all positive integers < 97
Approach 2: try all primes < 97
Approach 3: try all primes smaller than √97

Review questions

Finding the greatest common divisor of two numbers
Question: what is the gcd of 233 and 541

Approach 1: factorization and minimum of powers
Approach 2: Euclid algorithm

Review questions

Finding the greatest common divisor of two numbers
Question: what is the gcd of 233 and 541

Approach 1: factorization and minimum of powers
Approach 2: Euclid algorithm

Review questions

Congruencies
Question: is 3 and 7 congruent modulo 4?

3 mod 4=3
7 mod 4=3
Yes they are congruent.

Review questions

Congruencies
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Yes they are congruent.

Review questions

Countable sets
Is a finite set countable?
**Review questions**

**Countable sets**

Is a finite set countable?  
Yes.

What other sets are called countable?  
A set that has the same cardinality as the set of positive integers $\mathbb{Z}^+$

How to show the countability of infinite sets?  
Show that a bijection in between $\mathbb{Z}^+$ and the set exists

**Example:**  
- Assume $A = \{0, 2, 4, 6, \ldots\}$ set of even numbers. Is it countable?
Review questions

Example:

• Assume $A = \{0, 2, 4, 6, \ldots \}$ set of even numbers. Is it countable?
• Using the definition: Is there a bijective function $f: \mathbb{Z}^+ \rightarrow A$?

$\mathbb{Z}^+ = \{1, 2, 3, 4, \ldots \}$

• Define a function $f: x \rightarrow 2x - 2$ (an arithmetic progression)
  • $1 \rightarrow 2(1)-2 = 0$
  • $2 \rightarrow 2(2)-2 = 2$
  • $3 \rightarrow 2(3)-2 = 4 \ldots$

• one-to-one (why?) $2x - 2 = 2y - 2 \Rightarrow 2x = 2y \Rightarrow x = y$.
• onto (why?) $\forall a \in A, (a+2)/2$ is the pre-image in $\mathbb{Z}^+$.
• Therefore $|A| = |\mathbb{Z}^+|$.

Review questions

Countable sets

Is the set of real numbers countable?

No !!!

Review questions

Mathematical induction

• Used to prove statements of the form $\forall x \ P(x)$ where $x \in \mathbb{Z}$
• What are the two steps of the proof?

1) Basis step: The proposition $P(1)$ is true.
2) Inductive step: The implication $P(k) \rightarrow P(k+1)$, is true for all positive $k$.
• Therefore we conclude $\forall n \ P(n)$.
Mathematical induction

Example: Prove the sum of first $n$ odd integers is $n^2$.
  i.e. $1 + 3 + 5 + 7 + ... + (2n - 1) = n^2$ for all positive integers.

Proof:
  • What is $P(n)$? $P(n): 1 + 3 + 5 + 7 + ... + (2n - 1) = n^2$

Basic Step
  Show $P(1)$ is true
  • Trivial: $1 = 1^2$

Inductive Step
  Show if $P(n)$ is true then $P(n+1)$ is true for all $n$.
  • Suppose $P(n)$ be true, that is $1 + 3 + 5 + 7 + ... + (2n - 1) = n^2$
  • Show $P(n+1): 1 + 3 + 5 + 7 + ... + (2n + 1) = (n+1)^2$
    follows:
    $1 + 3 + 5 + 7 + ... + (2n - 1) + (2n + 1) = \frac{n^2}{n^2} + \frac{(2n+1)}{(2n+1)} = (n+1)^2$
Review questions

Mathematical induction

What is the difference between regular and strong induction?

• The regular induction:
  – uses the basic step \( P(1) \) and
  – inductive step \( P(k) \rightarrow P(k+1) \)

• Strong induction uses:
  – Uses the basis step \( P(1) \) and
  – inductive step \( P(1) \text{ and } P(2) \ldots P(k) \rightarrow P(k+1) \)

Recursive function:

A function on the set of nonnegative integers can be defined by

1. Specifying the value of the function at 0
2. Giving a rule for finding the function's value at \( n+1 \) in terms of the function's value at integers \( i \leq n \).

Example

Define the function:
\[ f(n) = 2n + 1 \quad n = 0, 1, 2, \ldots \]
recursively.

• \( f(0) = ? \)
• \( f(n+1) = ? \)

Example

Define the function:
\[ f(n) = 2n + 1 \quad n = 0, 1, 2, \ldots \]
recursively.

• \( f(0) = 1 \)
• \( f(n+1) = ? \)

Example:

Define the function:
\[ f(n) = 2n + 1 \quad n = 0, 1, 2, \ldots \]
recursively.

• \( f(0) = 1 \)
• \( f(n+1) = f(n) + 2 \)

Counting

Basic counting rules?
Review questions

Counting

Basic counting rules?
- Product rule
- Sum rule

How do we count with the product rule?

Example:
- How many different bit strings of length 7 are there?
  - E.g. 1011010
  - Is it possible to decompose the count problem and if yes how?
    - Yes.
      - Then count the number of possible assignments to bit 1
      - Count the number of possible assignments to bit 2
      - Total assignments to first 2 bits: \(2^2 = 4\)

Review

Example:
- How many different bit strings of length 7 are there?
  - E.g. 1011010
  - Is it possible to decompose the count problem and if yes how?
  - Yes.
    - Then count the number of possible assignments to bit 7
    - \(n = 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^7\)
Review questions

Counting

What is the pigeonhole principle?

• If there are \(k+1\) objects and \(k\) bins. Then there is at least one bin with two or more objects.

Theorem: If \(N\) objects are placed into \(k\) bins then there is at least one bin containing at least \(\lceil N/k \rceil\) objects.

Example:
There are 400 people. What can you say about their birthdays?
There are at least 2 people who have the same birthday.

Review questions

Counting

What are Permutations?

Ordered arrangements of \(n\) objects

Example: Assume \(S=\{A, B, C\}\)
Permutations: ?
Review questions

Counting

What are Permutations?
Ordered arrangements of n objects

Example: Assume $S=\{A, B, C\}$
Permutations:
ABC ACB BAC BCA CAB CBA
How many: $n!$
$3!=3\times2=6$

Review questions

Counting

What are $k$-combinations from set $S$?

• Unordered collections of $k$ elements from the set of $S$

Example: $S=\{A, B, C\}$
2-combinations:
AB AC BC
How many are there?
Review questions

Counting

What are k-combinations from set S?
• Unordered collections of k elements from the set of S

Example: \( S = \{A, B, C\} \)
2-combinations:
AB AC BC
How many are there?
• In our case: \( \frac{3!}{2!} = 3 \)

Example:
3 goalies, 8 defenders, 12 attackers on the hockey team
How many ways to put together the first line that includes:
One goalie
Two defenders
Three attackers

?