CS 441 Discrete Mathematics for CS
Lecture 18

Probabilities

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Course administration

- **Homework 8 : Due today**
- **No homework next week**
- **Midterm exam 2**
  - **Thursday, November 12, 2009**
  - **Covers only the material after midterm 1**
    - **Integers (Primes, Division, Congruencies)**
    - **Sequences and Summations**
    - **Inductive proofs and Recursion**
    - **Counting**

Course web page:

http://www.cs.pitt.edu/~milos/courses/cs441/

Probabilities

- **Experiment:**
  - a procedure that yields one of the possible outcomes
- **Sample space:** a set of possible outcomes
- **Event:** a subset of possible outcomes (E is a subset of S)
- **Assuming the outcomes are equally likely, the probability of an event E, defined by a subset of outcomes from the sample space S is**
  - **P(E)= |E| / |S|**

- The cardinality of the subset divided by the cardinality of the sample space.

Probabilities

- Event E, Sample space S, all outcomes equally likely, then

P(E)= |E| / |S|

Example:

- roll of two dices
- What is the probability that the outcome is 7.
- All possible outcomes (sample space S):
- (1,6) (2,6) …(6,1),…(6,6) total: 36

- Outcomes leading to 7 (event E)
- (1,6) (2,5) …(6,1) total: 6
- P(sum=7)= 6/36 = 1/6

Probabilities

- Event E, Sample space S, all outcomes equally likely, then

  P(E)= |E| / |S|

Example:

- Odd of winning a lottery: 6 numbers out of 40.
- Total number of outcomes (sample space S):
  - C(40,6) = 3,838,380
- Winning combination (event E): 1
- Probability of winning:
  - P(E) =1/C(40,6)= 34! 6! / 40! = 1/3,838,380

Probabilities

- Event E, Sample space S, all outcomes equally likely, then

  P(E)= |E| / |S|

Example (cont):

- Odd of winning a second prize in lottery: 6 numbers out of 40.
- Total number of outcomes (sample space S):
  - C(40,6) = 3,838,380
- Second prize (event E): C(6,5)*(40-6)= 6*34
- Probability of winning:
  - P(E) =6*34/C(40,6)= (6*34)/3,838,380

Probabilities

- Event E, Sample space S, all outcomes equally likely, then

  P(E)= |E| / |S|

Another lottery:

- 6 numbers with repetitions out of 40 numbers
- Total number of outcomes:
  - Permutations with repetitions: $= 40^6$
- Number of winning configuration: 1
  - P(win) = $1/40^6$

And its modification:

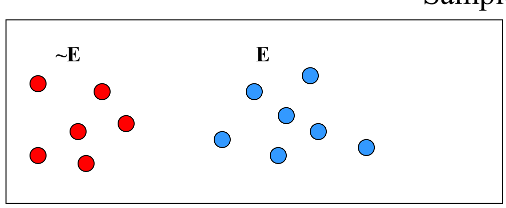
- If the winning combination is order independent:
  - E.g. (1,5,17,25,5,13) is equivalent to (5,17,5,1,25,13)
  - Number of winning permutations = number of permutations of 6 = 6!
  - P(win)= $6! / 40^6$

Probabilities

Theorem: Let E be an event and ~E its complement with regard to S. Then:

- P(~E) = 1 – P(E)

Sample space



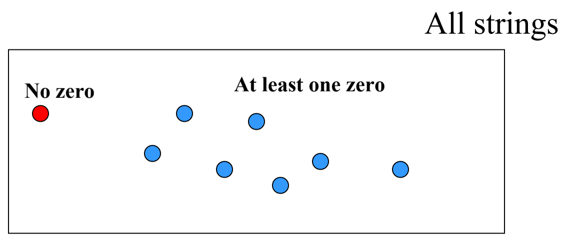
Proof.

P(~E) = (|S|- |E|) /| S | = 1- |E| / |S|

Probabilities

Example:

- 10 randomly generated bits. What is the probability that there is at least one zero in the string.



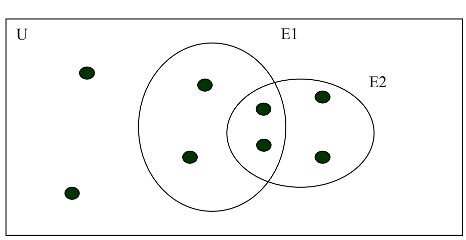
- Event: seeing no-zero string P(E) $=1/ 2^{10}$
- ~Event: seeing at least one zero in the string
  P(~E)= 1-P(E)=1- $1/2^{10}$

Probabilities

Theorem. Let E1 and E2 be two events in the sample space S. Then:

- P(E1 U E2) = P(E1) + P(E2) – P(E1 and E2)

- This is an example of the inclusion-exclusion principle



Probabilities

Theorem. Let E1 and E2 be two events in the sample space S. Then:

- P(E1 U E2) = P(E1) + P(E2) – P(E1 and E2)

Example: Probability that a positive integer <= 100 is divisible either by 2 or 5.

- P(E1)= 50/100
- P(E2)= 20/100
- P(E1 and E2) = 10/100
- P(E1 U E2) = (5+2-1)/10 = 6/10

Probabilities

- Assumption applied so far:
  - **the probabilities of each outcome are equally likely.**
- However in many cases outcome may not be equally likely.

Example: a biased coin or a biased dice.

- Probability of head 0.6, probability of a tail 0.4.

Probabilities

Three axioms of the probability theory:

- Probability of a discrete outcome is:
  - $0 <= P(s) <= 1$
- Sum of probabilities of all outcomes is = 1
- For any two events E1 and E2 holds:
  P(E1 U E2) = P(E1) + P(E2) – P(E1 and E2)

Probabilities

Definition: A function p: S → [0,1] satisfying the three conditions is called **a probability distribution**

Example: a biased coin

- Probability of head 0.6, probability of a tail 0.4
- **Probability distribution:**
  - Head → 0.6
  - Tail → 0.4

  The sum of the probabilities sums to 1

Note: **a uniform distribution** is a special distribution that assigns an equal probability to each outcome.

Conditional probability

Definition: Let E and F be two events such that P(F) > 0. The **conditional probability** of E given F

- P(E|F) = P(E and F) / P(F)

Example:

- What is the probability that a family has two boys given that they have at least one boy. Assume the probability of having a girl or a boy is equal.
- **Possibilities. BB BG GB GG**
- **P(BB) = ¼**
- **P(one boy) = ¾**
- **P(BB|given a boy) = ¼ / ¾ = 1/3**

Conditional probability

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BB BG GB GG

Conditional probability

Corrolary: Let E and F be two events such that P(F) > 0. Then:

- P(E and F) = P(E|F) P(F)

Proof:

- From the definition of the conditional probability:

  P(E|F) = P(E and F) / P(F)

  →

  P(E and F) = P(E|F) P(F)

Conditional probability

Corrolary: Let E and F be two events such that P(F) > 0. Then:

- P(E and F) = P(E|F) P(F)

Example:

- Assume the probability of getting a flu is 0.2
- Assume the probability of having a high fever given the flu: 0.9

What is the probability of getting a flu with fever?

P(flu and fever) = P(fever|flu)P(flu) = 0.9*0.2 = 0.18

- When is this useful?

Sometimes conditional probabilities are easier to estimate.

Bayes theorem

Definition: Let E and F be two events such that P(F) > 0. Then:

- $P(E|F) = P(F|E)P(E) / P(F)$

Proof:

$P(E|F) = P(F \text{ and } E) / P(F)$

$= P(F|E)P(E) / P(F)$

Idea: Simply switch the conditioning events.

Bayes theorem

Definition: Let E and F be two events such that P(F) > 0. Then:

- $P(E|F) = P(F|E)P(E) / P(F)$

Example:

- Assume the probability of getting a flu is 0.2
- Assume the probability of getting a fever is 0.3
- Assume the probability of having a high fever given the flu: 0.9
- What is the probability of having a flu given the fever?
- P(flu | fever) = P(fever|flu) P(flu) / P(fever) =

= 0.9 x 0.2/0.3 = 0.18/0.6 = 0.3

Independence

Definition: The events E and F are said to be **independent** if:

- P(E and F) = P(E)P(F)

Example. Assume that E denotes the family has three children of both sexes and F the fact that the family has at most one boy. Are E and F independent?

- All combos = {BBB, BBG, BGB, GBB,BGG,GBG,GGB,GGG} the number of elements = 8
- Both sexes = {BBG BGB GBB BGG GBG GGB} # = 6
- At most one boy = {GGG GGB GBG BGG} # = 4
- E and F = {GGB GBG BGG} # = 3
- P(E and F} = 3/8 and P(E)*P(F)= 4/8 6/8 = 3/8

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- E and F = {GGB GBG BGG} # = 3
- P(E and F} = 3/8 and P(E)*P(F)= 4/8 6/8 = 3/8
- **The two probabilities are equal → E and F are independent**