CS 441 Discrete Mathematics for CS
Lecture 16

Counting

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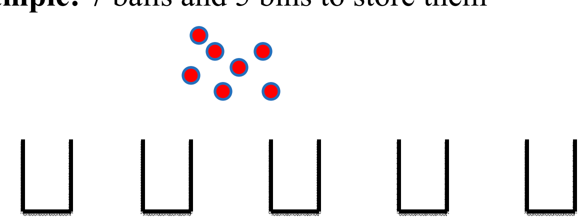
Pigeonhole principle

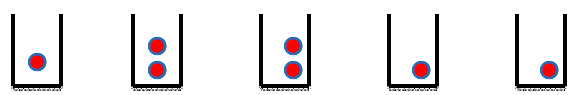
- Assume you have a set of objects and a set of bins used to store objects.
- **The pigeonhole principle** states that if there are more objects than bins then there is at least one bin with more than one object.
- **Example:** 7 balls and 5 bins to store them



Pigeonhole principle

- Assume you have a set of objects and a set of bins used to store objects.
- **The pigeonhole principle** states that if there are more objects than bins then there is at least one bin with more than one object.

- **Example:** 7 balls and 5 bins to store them
- At least one bin with more than 1 ball exists.



Pigeonhole principle

- **Theorem. If there are k+1 objects and k bins. Then there is at least one bin with two or more objects.**

Pigeonhole principle

Example:

- Assume 367 people. Are there any two people who have the same birthday?

- How many days are in the year? 365.
- Then there must be at least two people with the same birthday.

Generalized pigeonhole principle

Theorem. If N objects are placed into k bins then there is at least one bin containing at least $\lceil N/k \rceil$ objects.

Example. Assume 100 people. Can you tell something about the number of people born in the same month.

- Yes. There exists a month in which at least $\lceil 100/12 \rceil = \lceil 8.3 \rceil = 9$ people were born.

Permutations

A permutation of a set of distinct objects is an **ordered arrangement** of the objects. Since the objects are distinct, they cannot be selected more than once. Furthermore, the order of the arrangement matters.

Example:

- Assume we have a set S with n elements. S={a,b,c}.
- **Permutations of S:**
- **a b c   a c b   b a c   b c a   c a b   c b a**

Number of permutations

- Assume we have a set S with n elements. $S=\{a_1\ a_2\ \ldots\ a_n\}$.
- **Question:** How many different permutations are there?

- The number of permutations is

  $P(n,n) = n(n-1)(n-2)\ldots 2.1 = n!$

k-permutations

- ***k*-permutation** is an ordered arrangement of k elements of a set.

- The number of k-permutations of a set with n distinct elements is:

  P(n,k) = n(n-1)(n-2)…(n-k+1)= n!/(n-k)!

k-permutations

Example:

Suppose that there are eight runners in a race. The winner receives a gold medal, the second-place finisher receives a silver medal, and the third-place finisher receives a bronze medal. How many different ways are there to award these medals, if all possible outcomes of the race can occur?

Answer:

note that the runners are distinct and that the medals are ordered. The solution is P(8,3) = 8 * 7 * 6 = 8! / (8-3)! = 336.

Combinations

A *k*-combination of elements of a set is an **unordered** selection of k elements from the set. Thus, a k-combination is simply a subset of the set with k elements.

Example:

- 2-combinations of the set {a,b,c}:
- ?

Combinations

A *k*-combination of elements of a set is an **unordered** selection of k elements from the set. Thus, a k-combination is simply a subset of the set with k elements.

Example:

- 2-combinations of the set {a,b,c}

  a b   a c   b c

  a b   covers 2-permutations: **a b** and **b a**

Combinations

Theorem: The number of k-combinations of a set with n distinct elements, where n is a positive integer and k is an integer with $0 \leq k \leq n$ is

$$C(n,k) = \frac{n!}{(n-k)!\,k!}$$

Combinations

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Proof: The k-permutations of the set can be obtained by forming the $C(n,k)$ k-combinations of the set, and then ordering the elements in each k-combination, which can be done in $P(k,k)$ ways. Consequently,

$P(n,k) = C(n,k) * P(k,k)$.

This implies that

$$C(n,k) = P(n,k) / P(k,k) = P(n,k) / k! = \frac{n!}{(n-k)!\,k!}$$

Combinations

Proof (intuition). Assume elements a set {A1, A2, A3, A4, A5}. All 3-combinations of elements are:

- A1 A2 A3
- A1 A2 A4
- A1 A2 A5
- A1 A3 A4
- A1 A3 A5
- A1 A4 A5
- A2 A3 A4
- A2 A3 A5
- A2 A4 A5
- A3 A4 A5
- **Total of 10.**

Combinations

Intuition (example): Assume elements a set {A1, A2, A3, A4, A5}. All 3-combinations of elements are:

- A1 A2 A3
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- A2 A3 A5
- A2 A4 A5
- A3 A4 A5
- **Total of 10.**

Each combination leads to P(3,3) different 3-permutations

Combinations

Intuition (example): Assume elements A1, A2, A3, A4 and A5 in the set. All 3-combinations of elements are:

- A1 A2 A3
- A1 A2 A4
- A1 A2 A5
- A1 A3 A4
- A1 A3 A5
- A1 A4 A5
- A2 A3 A4
- A2 A3 A5
- A2 A4 A5
- A3 A4 A5
- **Total of 10.**

Each combination cover many 3-permutations

A1 A2 A3
A1 A3 A2
A2 A1 A3
A2 A3 A1
A3 A1 A2
A3 A2 A1

Combinations

Intuition (example): Assume elements A1, A2, A3, A4 and A5 in the set. All 3-combinations of elements are:

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- A1 A2 A4
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- A2 A3 A4
- A2 A3 A5
- A2 A4 A5
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- **Total of 10.**

Each 3-combination covers many 3-permutations

A1 A2 A3
A1 A3 A2
A2 A1 A3
A2 A3 A1
A3 A1 A2
A3 A2 A1

How many 3-permutations are there?

Combinations

Intuition (example): Assume elements A1, A2, A3, A4 and A5 in the set. All 3-combinations of elements are:

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- A2 A3 A5
- A2 A4 A5
- A3 A4 A5
- **Total of 10.**

Each 3-combination covers many 3-permutations

A1 A2 A3
A1 A3 A2
A2 A1 A3
A2 A3 A1
A3 A1 A2
A3 A2 A1

How many 3-permutations are there?

$P(3,3) = 3! = 6$

Combinations

Intuition (example): Assume elements A1, A2, A3, A4 and A5 in the set. All 3-combinations of elements are:

- A1 A2 A3
- A1 A2 A4
- A1 A2 A5
- A1 A3 A4
- A1 A3 A5
- A1 A4 A5
- A2 A3 A4
- A2 A3 A5
- A2 A4 A5
- A3 A4 A5
- **Total of 10.**

Each 3-combination covers many 3-permutations

So: $P(5,3) = C(5,3)\,P(3,3)$

Then: $C(5,3) = P(5,3)/P(3,3)$
$= 5! / (2!\,3!) = 10$

Combinations

Example:

- We need to create a team of 5 player for the competition out of 10 team members. How many different teams is it possible to create?

Answer:

- When creating a team we do not care about the order in which players were picked. It is is important that the player is in. Because of that we need to consider unordered sets of people.

- C(10,5) = 10!/(10-5)!5! = (10.9.8.7.6) / (5 4 3 2 1)
  = 2.3.2.7.3= 6.14.3= 6.42= **252**

Combinations

Corrolary:

- C(n,k) = C(n,n-k)

Proof.

- C(n,k) = n! / (n-k)! k!
  = n!/ (n-k)! (n – (n-k))!
  = C(n,n-k)

Binomial coefficients

- The number of k-combinations out of n elements, C(n,k), is often denoted as:

$$\binom{n}{k}$$

and reads **n choose k**. The number is also called **a binomial coefficient.**

- Binomial coefficients occur as coefficients in the expansion of powers of binomial expressions such as

$$(a+b)^n$$

- **Definition:** a binomial expression is the sum of two terms *(a+b).*

Binomial coefficients

Example:

- Expansion of the binomial expression $(a+b)^3$.

$$(a+b)^3 =$$

$$(a+b)(a+b)(a+b) =$$

$$(a^2+2ab+b^2)(a+b) =$$

$$a^3+2a^2b+ab^2+a^2b+2ab^2+b^3 =$$

$$1a^3+3a^2b+3ab^2+1b^3$$

1 3 3 1 Binomial coefficients

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

Binomial coefficients

Binomial theorem: Let a and b be variables and n be a nonnegative integer. Then:

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i$$

$$= \binom{n}{0} a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \ldots + \binom{n}{n-1} a^1 b^{n-1} + \binom{n}{n} b^n$$

Binomial coefficients

Binomial theorem: Let a and b be variables and n be a nonnegative integer. Then:

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i$$

Proof. The products after the expansion include terms $a^{(n-i)} b^i$ for all i=0,1, …n. To obtain the number of such coefficients note that we have to choose exactly (n-i) a(s) out of the product of n binomial expressions.

(n-i) picks

$$(a+b)^n = \underbrace{(a+b)(a+b)(a+b)\ldots(a+b)}_{n}$$

- The number of ways we pull a(s) out of the product is given as:

$$\binom{n}{n-i} = \binom{n}{i}$$

Binomial coefficients

Corrolary: Let n be a nonnegative integer. Then:

$$\sum_{i=0}^{n} \binom{n}{i} = 2^{n}$$

Proof:

- Assume a set with n elements:
- C(n,0) = number of subsets of size 0.
- C(n,i) = the number of subsets of size i.
- C(n,n) = the number of subsets of size n.
- The sum of these numbers must give me the number of all subsets of the set n.
- We know it is 2^n so the result follows.

Binomial coefficients

Corrolary:

- Let n be a nonnegative integer. Then:

$$\sum_{i=0}^{n} (-1)^{i} \binom{n}{i} = 0$$

Proof:

$$\sum_{i=0}^{n} \binom{n}{i} (-1)^{i} = \sum_{i=0}^{n} \binom{n}{i} (-1)^{i} 1^{n-i} = ((-1)+1)^{n} = 0^{n} = 0$$

Binomial coefficients

Example:

- Show that $$\sum_{i=0}^{n} \binom{n}{i} 2^i = 3^n$$

- **Answer:**

$$\sum_{i=0}^{n} \binom{n}{i} 2^i = \sum_{i=0}^{n} \binom{n}{i} (2)^i 1^{n-i} = (2+1)^n = 3^n$$

Binomial coefficients

Question: We have binomial coefficients for expressions with the power n. Are binomial coefficients for powers of n-1 or n+1 in any way related to coefficients for n ?

- **The answer is yes.**

Theorem:

- Let n and k be two positive integers with . Then it holds:

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

Pascal traingle

Drawing the binomial coefficients for different powers in increasing order gives a Pascal triangle:

	powers
1	
1 1	
1 2 1	2
1 3 3 1	3
1 4 6 4 1	4
1 5 10 10 5 1	5
1 6 15 20 15 6 1	6
…	7