Counting

• Assume we have a set of objects with certain properties
• Counting is used to determine the number of these objects

Examples:
• Number of available phone numbers with 7 digits in the local calling area
• Number of possible match starters (football, basketball) given the number of team members and their positions
Basic counting rules

- Counting problems may be hard, and easy solution not obvious
- **Approach:**
  - simplify the solution by decomposing the problem

- **Two basic decomposition rules:**
  - **Product rule**
    - A count decomposes into a sequence of dependent counts
      ("each element in the first count is associated with all elements of the second count")
  - **Sum rule**
    - A count decomposes into a set of independent counts
      ("elements of counts are alternatives")

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Product rule

- **Product rule:** If a count of elements can be broken down into a sequence of dependent counts where the first count yields $n_1$ elements, the second $n_2$ elements, and kth count $n_k$ elements, by the product rule the total number of elements is:
  - $n = n_1 \times n_2 \times \ldots \times n_k$

**Example:** assume an auditorium with a seat labeled by a letter and numbers in between 1 to 50 (e.g. A23). We want the total number of seats in the auditorium.
- 26 letters and 50 numbers
- Number of seats: 26*50
### Product rule

**Example:**
- How many different bit strings of length 7 are there?
  - E.g. 1011010
- Assume the following decomposition:

  ![Diagram](image)

  Total assignments to 7 bits: $2^7$

### Sum rule

A count decomposes into a set of independent counts

**Example:**
- You need to travel in between city A and B. You can either fly, take a train, or a bus. There are 12 different flights in between A and B, 5 different trains and 10 buses. How many options do you have to get from A to B?
- We can take only one type of transportation and for each only one option. The number of options:
  - $n = 12 + 5 + 10$

**Sum rule:**
- $n = \text{number of flights} + \text{number of trains} + \text{number of buses}$
Beyond basic counting rules

**Example:** A login password.
- The minimum password length is 6 and the maximum is 8. The password can consist of either an uppercase letter or a digit. There must be at least one digit in the password.

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Inclusion-Exclusion principle

*Used in counts where the decomposition yields two count tasks with overlapping elements*
- If we used the sum rule some elements would be counted twice

**Inclusion-exclusion principle:** uses a sum rule and then corrects for the overlapping elements.

We used the principle for the cardinality of the set union.
- $|A \cup B| = |A| + |B| - |A \cap B|$
Inclusion-exclusion principle

**Example:** How many bitstrings of length 8 start either with a bit 1 or end with 00?

- **Count strings that start with 1:**
  - How many are there? $2^7$
- **Count the strings that end with 00.**
  - How many are there? $2^6$
- **The two counts overlap !!!**
  - How many of strings were counted twice? $2^5$ (1 x x x x x 0 0)

- Thus we can correct for the overlap simply by using:
  - $2^7 + 2^6 - 2^5 = 128 + 64 - 32 = 160$

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Pigeonhole principle

- Assume you have a set of objects and a set of bins used to store objects.
- **The pigeonhole principle** states that if there are more objects than bins then there is at least one bin with more than one object.

- **Example:** 7 balls and 5 bins to store them

![Diagram of 7 balls and 5 bins]
Pigeonhole principle

• Assume you have a set of objects and a set of bins used to store objects.

• The pigeonhole principle states that if there are more objects than bins then there is at least one bin with more than one object.

• Example: 7 balls and 5 bins to store them
  • At least one bin with more than 1 ball exists.

Pigeonhole principle

• **Theorem.** If there are k+1 objects and k bins. Then there is at least one bin with two or more objects.

• **Proof (by contradiction)**
  • Assume that we have k + 1 objects and every bin has at most one element. Then the total number of elements is k which is a contradiction.
  • End of proof
Pigeonhole principle

Example:
- Assume 367 people. Are there any two people who have the same birthday?
  - How many days are in the year? 365.
  - Then there must be at least two people with the same birthday.

Generalized pigeonhole principle
- We can often say more about the number of objects.
- Say we have 5 bins and 12 objects. What is it we can say about the bins and number of elements they hold?
  - There must be a bin with at least 3 elements.
  - Why?
    - Assume there is no bin with more than 3 elements.
    - Then the max number of elements we can have in 5 bins is 10. Since we need to place 12 objects at least one bin should have at least 3 elements.
**Generalized pigeonhole principle**

**Theorem.** If \( N \) objects are placed into \( k \) bins then there is at least one bin containing at least \( \lceil N / k \rceil \) objects.

**Example.** Assume 100 people. Can you tell something about the number of people born in the same month.

- Yes. There exists a month in which at least \( \lceil 100 / 12 \rceil = \lceil 8.3 \rceil = 9 \) people were born.

**Example.** Show that among any set of 5 integers, there are 2 with the same remainder when divided by 4.

**Answer:**
- Let there be 4 boxes, one for each remainder when divided by 4.
- After 5 integers are sorted into the boxes, there are \( \lceil 5/4 \rceil = 2 \) in one box.
Generalized pigeonhole principle

Example:
• How many students, each of whom comes from one of the 50 states, must be enrolled in a university to guaranteed that there are at least 100 who come from the same state?

Answer:
• Let there by 50 boxes, one per state.
• We want to find the minimal $N$ so that $\lceil N/50 \rceil=100$.
• Letting $N=5000$ is too much, since the remainder is 0.
• We want a remainder of 1 so that let $N=50\times99+1=4951$.

Permutations

A permutation of a set of distinct objects is an ordered arrangement of the objects. Since the objects are distinct, they cannot be selected more than once. Furthermore, the order of the arrangement matters.

Example:
• Assume we have a set $S$ with $n$ elements. $S=\{a,b,c\}$.
• Permutations of $S$:
  • $a\ b\ c\ a\ c\ b\ b\ a\ c\ a\ b\ c\ a$
**Number of permutations**

• Assume we have a set S with n elements. \( S = \{a_1, a_2, \ldots, a_n\} \).

• **Question:** How many different permutations are there?

• In how many different ways we can choose the first element of the permutation? \( n \) (either \( a_1 \) or \( a_2 \) … or \( a_n \))

• Assume we picked \( a_2 \).

• In how many different ways we can choose the remaining elements? \( n-1 \) (either \( a_1 \) or \( a_3 \) … or \( a_n \) but not \( a_2 \))

• Assume we picked \( a_j \).

• In how many different ways we can choose the remaining elements? \( n-2 \) (either \( a_1 \) or \( a_3 \) … or \( a_n \) but not \( a_2 \) and not \( a_j \))

\[
P(n, n) = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 1 = n!
\]

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**Permutations**

**Example 1.**

• How many permutations of letters \{a,b,c\} are there?

• Number of permutations is:

\[
P(n, n) = P(3, 3) = 3! = 6
\]

• Verify:

\[
abc \quad acb \quad bac \quad bca \quad cab \quad cba
\]
Permutations

Example 2
• How many permutations of letters A B C D E F G H contain a substring ABC.

Idea: consider ABC as one element and D, E, F, G, H as other 5 elements for the total of 6 elements.

Then we need to count the number of permutation of these elements.

$6! = 720$

$k$-permutations

• $k$-permutation is an ordered arrangement of $k$ elements of a set.
• The number of $k$-permutations of a set with $n$ distinct elements is:

$$P(n,k) = n(n-1)(n-2)\ldots(n-k+1) = \frac{n!}{(n-k)!}$$
**k-permutations**

- **k-permutation** is an ordered arrangement of \( k \) elements of a set.
- The number of \( k \)-permutations of a set with \( n \) distinct elements is:

\[
P(n,k) = \frac{n(n-1)(n-2)\ldots(n-k+1)}{(n-k)!} = \frac{n!}{(n-k)!}
\]

**Explanation:**
- Assume we have a set \( S \) with \( n \) elements. \( S = \{a_1, a_2, \ldots, a_n\} \).
- The 1st element of the \( k \)-permutation may be any of the \( n \) elements in the set.
- The 2nd element of the \( k \)-permutation may be any of the \( n-1 \) remaining elements of the set.
- And so on. For last element of the \( k \)-permutation, there are \( n-k+1 \) elements remaining to choose from.

**Example:**

The 2-permutations of set \( \{a,b,c\} \) are:

\( ab, ac, ba, bc, ca, cb \).

The number of 2-permutations of this 3-element set is

\[
P(n,k) = P(3,2) = 3(3-2+1) = 6.
\]
**k-permutations**

**Example:**
Suppose that there are eight runners in a race. The winner receives a gold medal, the second-place finisher receives a silver medal, and the third-place finisher receives a bronze medal. How many different ways are there to award these medals, if all possible outcomes of the race can occur?

**Answer:**
Note that the runners are distinct and that the medals are ordered.
The solution is $P(8,3) = 8 \cdot 7 \cdot 6 = 8! / (8-3)! = 336$. 