CS 441 Discrete Mathematics for CS
Lecture 15

Counting

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Counting

- Assume we have a set of **objects with certain properties**
- **Counting** is used to determine **the number of these objects**

Examples:

- Number of available phone numbers with 7 digits in the local calling area
- Number of possible match starters (football, basketball) given the number of team members and their positions

Basic counting rules

- Counting problems may be hard, and easy solution not obvious
- **Approach:**
  - **simplify the solution by decomposing the problem**

- **Two basic decomposition rules:**
  - **Product rule**
    - A count decomposes into a sequence of dependent counts ("each element in the first count is associated with all elements of the second count")
  - **Sum rule**
    - A count decomposes into a set of independent counts ("elements of counts are alternatives")

Product rule

- **Product rule:** If a count of elements can be broken down into a sequence of dependent counts where the first count yields $n1$ elements, the second $n2$ elements, and kth count nk elements, by the product rule the total number of elements is:
  - $n = n1*n2* \ldots * nk$

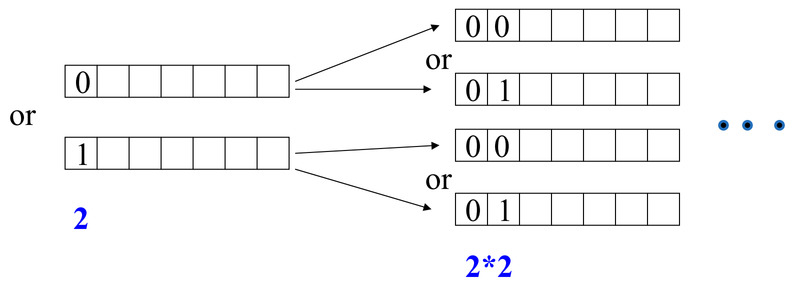
Example: assume an auditorium with a seat labeled by a letter and numbers in between 1 to 50 (e.g. A23). We want the total number of seats in the auditorium.

- 26 letters and 50 numbers
- Number of seats: 26*50

Product rule

Example:

- How many different bit strings of length 7 are there?
  - E.g. 1011010
- Assume the following decomposition:



Total assignments to 7 bits: 2^7

Sum rule

A count decomposes into a set of independent counts

Example:

- You need to travel in between city A and B. You can either fly, take a train, or a bus. There are 12 different flights in between A and B, 5 different trains and 10 buses. How many options do you have to get from A to B?
- We can take only one type of transportation and for each only one option. The number of options:
  - n = 12+5+10

Sum rule:

- n = number of flights + number of trains + number of buses

Beyond basic counting rules

Example: A login password.

- The minimum password length is 6 and the maximum is 8. The password can consist of either an uppercase letter or a digit. There must be at least one digit in the password.

Inclusion-Exclusion principle

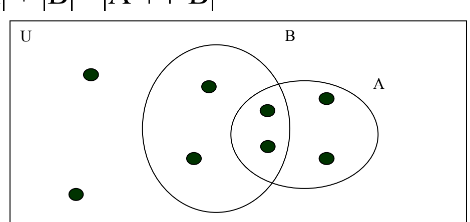
Used in counts where the decomposition yields two count tasks with overlapping elements

- If we used the sum rule some elements would be counted twice

Inclusion-exclusion principle: uses a sum rule and then corrects for the overlapping elements.

We used the principle for the cardinality of the set union.

- $|A \cup B| = |A| + |B| - |A \cap B|$



Inclusion-exclusion principle

Example: How many bitstrings of length 8 start either with a bit 1 or end with 00?

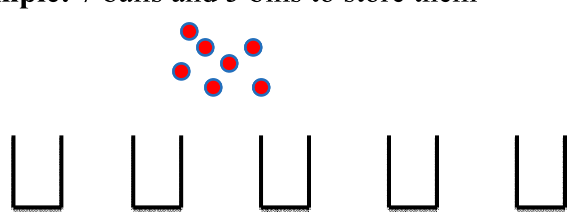
- **Count strings that start with 1**:
- How many are there? 2^7
- **Count the strings that end with 00**.
- How many are there? 2^6
- **The two counts overlap !!!**
- How many of strings were counted twice? 2^5 (1 xxxxx 00)

- Thus we can correct for the overlap simply by using:
- $2^7 + 2^6 - 2^5 = 128 + 64 - 32 =$ **160**

Pigeonhole principle

- Assume you have a set of objects and a set of bins used to store objects.
- **The pigeonhole principle** states that if there are more objects than bins then there is at least one bin with more than one object.

- **Example:** 7 balls and 5 bins to store them



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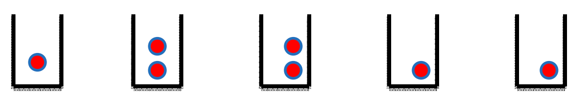
- **Example:** 7 balls and 5 bins to store them
- At least one bin with more than 1 ball exists.



Pigeonhole principle

- **Theorem. If there are k+1 objects and k bins. Then there is at least one bin with two or more objects.**

- **Proof (by contradiction)**
- Assume that we have k + 1 objects and every bin has at most one element. Then the total number of elements is k which is a contradiction.
- End of proof

Pigeonhole principle

Example:

- Assume 367 people. Are there any two people who have the same birthday?
- How many days are in the year? 365.
- Then there must be at least two people with the same birthday.

Generalized pigeonhole principle

- We can often say more about the number of objects.
- Say we have 5 bins and 12 objects. What is it we can say about the bins and number of elements they hold?
- There must be **a bin with at least 3 elements**.
- Why?
- Assume there is no bin with more than 3 elements.
- Then the max number of elements we can have in 5 bins is 10. Since we need to place 12 objects at least one bin should have at least 3 elements.

Generalized pigeonhole principle

Theorem. If N objects are placed into k bins then there is at least one bin containing at least $\lceil N/k \rceil$ objects.

Example. Assume 100 people. Can you tell something about the number of people born in the same month.

- Yes. There exists a month in which at least $\lceil 100/12 \rceil = \lceil 8.3 \rceil = 9$ people were born.

Generalized pigeonhole principle

Example.

- Show that among any set of 5 integers, there are 2 with the same remainder when divided by 4.

Answer:

- Let there be 4 boxes, one for each remainder when divided by 4.
- After 5 integers are sorted into the boxes, there are $\lceil 5/4 \rceil = 2$ in one box.

Generalized pigeonhole principle

Example:

- How many students, each of whom comes from one of the 50 states, must be enrolled in a university to guaranteed that there are at least 100 who come from the same state?

Answer:

- Let there by 50 boxes, one per state.
- We want to find the minimal N so that $\lceil N/50 \rceil=100$.
- Letting N=5000 is too much, since the remainder is 0.
- We want a remainder of 1 so that let N=50*99+1=4951.

Permutations

A permutation of a set of distinct objects is an **ordered arrangement** of the objects. Since the objects are distinct, they cannot be selected more than once. Furthermore, the order of the arrangement matters.

Example:

- Assume we have a set S with n elements. S={a,b,c}.
- **Permutations of S:**
- **a b c   a c b   b a c   b c a   c a b   c b a**

Number of permutations

- Assume we have a set S with n elements. $S=\{a_1\ a_2 \dots a_n\}$.
- **Question:** How many different permutations are there?

- In how many different ways we can choose the first element of the permutation? **n** (**either** a_1 or a_2 … or a_n)
- Assume we picked a_2.
- In how many different ways we can choose the remaining elements? **n-1** (**either** a_1 or a_3 … or a_n **but not** a_2**)**
- **Assume** we picked a_j.
- In how many different ways we can choose the remaining elements? **n-2** (**either** a_1 or a_3 … or a_n **but not** a_2 **and not** a_j)

  P(n,n) = n.(n-1)(n-2)…1 = n!

Permutations

Example 1.

- How many permutations of letters {a,b,c} are there?
- Number of permutations is:

  P(n,n) =P(3,3) =3! = 6
- Verify:

  abc acb bac bca cab cba

Permutations

Example 2

- How many permutations of letters A B C D E F G H contain a substring ABC.

Idea: consider ABC as one element and D,E,F,G,H as other 5 elements for the total of 6 elements.

Then we need to count the number of permutation of these elements.

$6! = 720$

k-permutations

- ***k*-permutation** is an ordered arrangement of *k* elements of a set.
- The number of *k*-permutations of a set with *n* distinct elements is:

$$P(n,k) = n(n-1)(n-2)\ldots(n-k+1) = n!/(n-k)!$$

k-permutations

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- The number of k-permutations of a set with n distinct elements is:

  P(n,k) = n(n-1)(n-2)…(n-k+1)= n!/(n-k)!

Explanation:

- Assume we have a set S with n elements. $S=\{a_1\ a_2\ \ldots\ a_n\}$.
- The 1st element of the k-permutation may be any of the n elements in the set.
- The 2nd element of the k-permutation may be any of the $n-1$ remaining elements of the set.
- And so on. For last element of the k-permutation, there are $n-k+1$ elements remaining to choose from.

k-permutations

Example:

The 2-permutations of set $\{a,b,c\}$ are:

ab, ac, ba, bc, ca, cb.

The number of 2-permutations of this 3-element set is

$P(n,k) = P(3,2) = 3\ (3\text{-}2\text{+}1) = 6.$

k-permutations

Example:

Suppose that there are eight runners in a race. The winner receives a gold medal, the second-place finisher receives a silver medal, and the third-place finisher receives a bronze medal. How many different ways are there to award these medals, if all possible outcomes of the race can occur?

Answer:

note that the runners are distinct and that the medals are ordered. The solution is P(8,3) = 8 * 7 * 6 = 8! / (8-3)! = 336.