CS 441 Discrete Mathematics for CS
Lecture 13

Mathematical induction & Recursion

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Proofs

Basic proof methods:

- Direct, Indirect, Contradiction, By Cases, Equivalences

Proof of quantified statements:

- **There exists x with some property P(x).**
  - It is sufficient to find one element for which the property holds.
- **For all x some property P(x) holds.**
  - Proofs of 'For all x some property P(x) holds' must cover all x and can be harder.
- **Mathematical induction** is a technique that can be applied to prove the universal statements for sets of positive integers or their associated sequences.

Mathematical induction

- Used to prove statements of the form $\forall x\, P(x)$ where $x \in Z^+$

Mathematical induction proofs consists of two steps:

1) **Basis:** The proposition P(1) is true.
2) **Inductive Step:** The implication

   $P(n) \rightarrow P(n+1)$, is true for all positive n.

- Therefore we conclude $\forall x\, P(x)$.

- **Based on the well-ordering property:** Every nonempty set of nonnegative integers has a **least element**.

Mathematical induction

Example: Prove the sum of first n odd integers is n^2 .

i.e. $1 + 3 + 5 + 7 + ... + (2n - 1) = n^2$ for all positive integers.

Proof:

- What is P(n)? P(n): $1 + 3 + 5 + 7 + ... + (2n - 1) = n^2$

Basis Step Show P(1) is true

- Trivial: $1 = 1^2$

Inductive Step Show if P(n) is true then P(n+1) is true for all n.

- Suppose P(n) be true, that is $1 + 3 + 5 + 7 + ... + (2n - 1) = n^2$
- Show P(n+1): $1 + 3 + 5 + 7 + ... + (2n - 1) + (2n + 1) =(n+1)^2$ follows:
- $\underbrace{1 + 3 + 5 + 7 + ... + (2n - 1)}_{n^2} + (2n + 1) =$

  $n^2 + (2n+1) = (n+1)^2$

Correctness of the mathematical induction

Suppose **P(1) is true** and **P(n) $\rightarrow$ P(n+1) is true** for all positive integers n. Want to show **$\forall$x P(x).**

Assume there is at least one n such that P(n) is false. Let S be the set of nonnegative integers where P(n) is false. Thus $S \neq \emptyset$.

Well-Ordering Property: Every nonempty set of nonnegative integers has a least element.

By the Well-Ordering Property, S has a least member, say k. k > 1, since P(1) is true. This implies k - 1 > 0 and P(k-1) is true (since remember k is the smallest integer where P(k) is false).

Now: P(k-1) $\rightarrow$ P(k) is true

thus, P(k) must be true (a contradiction).

- **Therefore $\forall$x P(x).**

Mathematical induction

Example: Prove $n < 2^n$ for all positive integers n.

- P(n): $n < 2^n$

Basis Step: $1 < 2^1$ (obvious)

Inductive Step: If P(n) is true then P(n+1) is true for each n.

- Suppose P(n): $n < 2^n$ is true
- Show P(n+1): $n+1 < 2^{n+1}$ is true.

$$\begin{aligned} n + 1 &< 2^n + 1 \\ &< 2^n + 2^n \\ &= 2^n(1 + 1) \\ &= 2^n (2) \\ &= 2^{n+1} \end{aligned}$$

Mathematical induction

Example: Prove $n^3 - n$ is divisible by 3 for all positive integers.

- $P(n)$: $n^3 - n$ is divisible by 3

Basis Step: $P(1)$: $1^3 - 1 = 0$ is divisible by 3 (obvious)

Inductive Step: If $P(n)$ is true then $P(n+1)$ is true for each positive integer.

- Suppose $P(n)$: $n^3 - n$ is divisible by 3 is true.
- Show $P(n+1)$: $(n+1)^3 - (n+1)$ is divisible by 3.

$$(n+1)^3 - (n+1) = n^3 + 3n^2 + 3n + 1 - n - 1$$
$$= (n^3 - n) + 3n^2 + 3n$$
$$= (n^3 - n) + 3(n^2 + n)$$

divisible by 3   divisible by 3

Strong induction

- **The regular induction:**
  - uses the basic step **P(1)** and
  - inductive step **P(n-1) → P(n)**
- **Strong induction uses:**
  - Uses the basis step **P(1)** and
  - inductive step **P(1) and P(2) … P(n-1) → P(n)**

Example: Show that a positive integer greater than 1 can be written as a product of primes.

Strong induction

Example: Show that a positive integer greater than 1 can be written as a product of primes.

Assume P(n): an integer n can be written as a product of primes.

Basis step: P(2) is true

Inductive step: Assume true for P(2),P(3), … P(n)

Show that P(n+1) is true as well.

2 Cases:

- If n+1 is a prime then P(n+1) is trivially true
- If n+1 is a composite then it can be written as a product of two integers (n+1) = a*b such that $1 < a, b < n+1$
- From the assumption P(a) and P(b) holds.
- Thus, n+1 can be written as a product of primes
- **End of proof**

Recursive Definitions

- Sometimes it is difficult to define an object explicitly, however it may be easy to define the object in terms of itself. This process is called **recursion.**

Examples of recursive definitions:

- Recursive definition of a geometric sequence:
  - $x_n = 3^n$
  - $x_0 = 1; \; x_n = 3x_{n-1}$
- Algorithm for computing the gcd:
  - **gcd(79, 35) = gcd(35, 9)**

Recursively Defined Functions

To define a function on the set of nonnegative integers

- 1. Specify the value of the function at 0
- 2. Give a rule for finding the function's value at n+1 in terms of the function's value at integers $i \le n$.

- **Such a definition is called recursive or inductive.**

Recursively defined functions

Example: Assume a recursive function on positive integers:

- $f(0) = 3$
- $f(n+1) = 2f(n) + 3$

- **What is the value of f(0) ?** 3
- $f(1) = 2f(0) + 3 = 2(3) + 3 = 6 + 3 = 9$
- $f(2) = f(1 + 1) = 2f(1) + 3 = 2(9) + 3 = 18 + 3 = 21$
- $f(3) = f(2 + 1) = 2f(2) + 3 = 2(21) = 42 + 3 = 45$
- $f(4) = f(3 + 1) = 2f(3) + 3 = 2(45) + 3 = 90 + 3 = 93$

Recursive definitions

- **Example:**
  Define the function:
  $f(n) = 2n + 1 \quad n = 0, 1, 2, \ldots$
  recursively.

- $f(0) = 1$
- $f(n+1) = f(n) + 2$

Recursive definitions

- **Example:**
  Define the sequence:
  $a_n = n^2 \quad \text{for } n = 1,2,3, \ldots$
  recursively.

- $a_1 = 1$
- $a_{n+1} = a_n^2 + (2n + 1), \quad n \geq 1$

Recursive definitions

- **Example:**
  Define a recursive definition of the sum of the first n positive integers:

$$F(n) = \sum_{i=1}^{n} i$$

- F(1) = 1
- F(n+1) = F(n) + (n+1) , $n \geq 1$

Recursive definitions

Some important functions or sequences in mathematics are defined recursively

Factorials

- n! = 1 if n=1
- n! = n.(n-1)! if $n \geq 1$

Fibonacci numbers:

- F(0)=0, F(1) =1 and
- F(n) =F(n-1) + F(n-2) for n=2,3, …

Recursive definitions

- **Greatest common divisor**

  gcd(a,b) = b if b | a

  = gcd(b, a mod b)

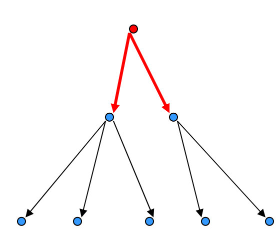
Recursive definitions

- **Data structures**

Example: Rooted tree

- **A basis step:**
  – a single node (vertex) is a rooted tree

- **Recursive step:**
  – Assume T1, T2, … Tk are rooted trees, then the graph with a root r connected to T1, T2, … Tk is a rooted tree



Recursive definitions

- **Assume the alphabet Σ**
  - Example: $\Sigma = \{a,b,c,d\}$
- **A set of all strings containing symbols in Σ: Σ^***
  - Example: $\Sigma^* = \{$"",a,aa,aaa,aaa…, ab, …b,bb, bbb, …$\}$

Recursive definition of Σ^*

- **Basis step:**
  - empty string $\lambda \in \Sigma^*$
- **Recursive step:**
  - If $w \in \Sigma^*$ and $x \in \Sigma$ then $wx \in \Sigma^*$