Sequences and summations

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Course administrivia

Weekly homework assignments
• Assigned in class and posted on the course web page
• Due one week later at the beginning of the lecture
• No extension policy

Collaboration policy:
• You may discuss the material covered in the course with your fellow students in order to understand it better
• However, homework assignments should be worked on and written up individually
Course administration

Midterm:
• Tuesday, October 6, 2009
• Closed book, in-class
• Covers Chapters 1 and 2.1-2.3 of the textbook

No Homework assignment this week

Course web page:
http://www.cs.pitt.edu/~milos/courses/cs441/

Midterm

• Propositional logic
  – Syntax/Logical connectives
  – Truth values/tables
  – Translation of English sentences
  – Equivalences
• Predicate logic
  – Syntax, quantified sentences
  – Truth values for sentences in predicate logic
  – Translations
  – Rules of inference
Midterm

• **Proofs**
  – Formal proofs
  – Informal proofs
  – Types of proofs: direct, indirect, contradiction …

• **Sets**
  – Basics: Set subsets, power set …
  – Cardinality of the set
  – N-tuples
  – Cartesian products
  – Set operators
  – Representation of sets

Midterm

• **Functions**
  – Basic definition
  – Function properties: injection, surjection, bijection
  – Function inverse
  – Composition of functions
Midterm

• Types of problems on midterm:
  – Knowledge of definitions, concepts, methods
    • E.g. what is a proposition, what is a set
  – Problems similar to homework assignments and exercises
    • E.g. prove is n is even than 3n+2 is even
  – If needed you will receive a list of logical equivalences
    and/or a list of inference rules

Sequences

**Definition:** A sequence is a function from a subset of the set of integers (typically \{0,1,2,...\} or \{1,2,3,...\}) to a set S. We use the notation \(a_n\) to denote the image of the integer \(n\). We call \(a_n\) a term of the sequence.

**Notation:** \(\{a_n\}\) is used to represent the sequence (note \{\} is the same notation used for sets, so be careful). \(\{a_n\}\) represents the ordered list \(a_1, a_2, a_3, \ldots\).

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & \ldots \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
1 & a_2 & a_3 & a_4 & a_5 & a_6 & \ldots \\
\end{array}
\]

\(\{a_n\}\)
Sequences

Examples:

• (1) \( a_n = n^2 \), where \( n = 1, 2, 3, \ldots \)
  – What are the elements of the sequence?
    1, 4, 9, 16, 25, ...

• (2) \( a_n = (-1)^n \), where \( n = 0, 1, 2, 3, \ldots \)
  – Elements of the sequence?
    1, -1, 1, -1, 1, ...

• (3) \( a_n = 2^n \), where \( n = 0, 1, 2, 3, \ldots \)
  – Elements of the sequence?
    1, 2, 4, 8, 16, 32, ...

Arithmetic progression

Definition: An arithmetic progression is a sequence of the form
\[ a, a+d, a+2d, \ldots, a+nd \]
where \( a \) is the initial term and \( d \) is common difference, such that both belong to \( \mathbb{R} \).

Example:

• \( s_n = -1+4n \) for \( n = 0, 1, 2, 3, \ldots \)
• members: -1, 3, 7, 11, …
Geometric progression

**Definition** A geometric progression is a sequence of the form:

\[ a, ar, ar^2, \ldots, ar^k, \]

where \( a \) is the *initial term*, and \( r \) is the *common ratio*. Both \( a \) and \( r \) belong to \( \mathbb{R} \).

**Example:**

- \( a_n = \left( \frac{1}{2} \right)^n \) for \( n = 0, 1, 2, 3, \ldots \)
  - members: 1, \( \frac{1}{2} \), \( \frac{1}{4} \), \( \frac{1}{8} \), \ldots

Sequences

- Given a sequence finding a rule for generating the sequence is not always straightforward

**Example:**

- Assume the sequence: 1, 3, 5, 7, 9, \ldots
- What is the formula for the sequence?
- Each term is obtained by adding 2 to the previous term.
- 1, 1+2=3, 3+2=5, 5+2=7
- It suggests an arithmetic progression: \( a+nd \)
  - with \( a=1 \) and \( d=2 \)
    - \( a_n=1+2n \) or \( a_n=1+2n \)
Sequences

• Given a sequence finding a rule for generating the sequence is not always straightforward

Example 2:
• Assume the sequence: 1, 1/3, 1/9, 1/27, …
• What is the sequence?
• The denominators are powers of 3.
  1, 1/3= 1/3, (1/3)/3=1/(3*3)=1/9, (1/9)/3=1/27
• What type of progression this suggests?

Sequences

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Example 2:
• Assume the sequence: 1, 1/3, 1/9, 1/27, …
• What is the sequence?
• The denominators are powers of 3.
  1, 1/3= 1/3, (1/3)/3=1/(3*3)=1/9, (1/9)/3=1/27
• This suggests a geometric progression: \( ar^k \)
  with \( a=1 \) and \( r=1/3 \)
  \( (1/3)^n \)
Summations

Summation of the terms of a sequence:

\[ \sum_{j=m}^{n} a_j = a_m + a_{m+1} + \ldots + a_n \]

The variable \( j \) is referred to as the index of summation.
- \( m \) is the lower limit and
- \( n \) is the upper limit of the summation.

Example:

1) Sum the first 7 terms of \( \{n^2\} \) where \( n=1,2,3, \ldots \).

\[ \sum_{j=1}^{7} j^2 = 1 + 4 + 9 + 16 + 25 + 36 + 49 = 140 \]

2) What is the value of

\[ \sum_{k=4}^{8} (-1)^k = (-1)^4 + (-1)^5 + (-1)^6 + (-1)^7 + (-1)^8 = 1 \]
Arithmetic series

**Definition:** The sum of the terms of the arithmetic progression $a, a+d, a+2d, \ldots, a+nd$ is called an arithmetic series.

**Theorem:** The sum of the terms of the arithmetic progression $a, a+d, a+2d, \ldots, a+nd$ is

$$S = \sum_{j=1}^{n} (a + jd) = na + d \sum_{j=1}^{n} j = na + d \frac{n(n+1)}{2}$$

- Why?

**Proof:**

$$S = \sum_{j=1}^{n} (a + jd) = \sum_{j=1}^{n} a + \sum_{j=1}^{n} jd = na + d \sum_{j=1}^{n} j$$
Arithmetic series

**Theorem:** The sum of the terms of the arithmetic progression
a, a+d,a+2d, …, a+nd is

\[ S = \sum_{j=1}^{n} (a + jd) = na + d \sum_{j=1}^{n} j = na + d \frac{n(n+1)}{2} \]

**Proof:**

\[ S = \sum_{j=1}^{n} (a + jd) = \sum_{j=1}^{n} a + \sum_{j=1}^{n} jd = na + d \sum_{j=1}^{n} j \]

\[ \sum_{j=1}^{n} j = 1 + 2 + 3 + 4 + \ldots + (n-2) + (n-1) + n \]

\[ 1 + (n-1) = n \]
Arithmetic series

**Theorem:** The sum of the terms of the arithmetic progression $a, a+d, a+2d, \ldots, a+nd$ is

$$S = \sum_{j=1}^{n} (a + jd) = na + d \sum_{j=1}^{n} j = na + d \frac{n(n+1)}{2}$$

**Proof:**

$$S = \sum_{j=1}^{n} (a + jd) = \sum_{j=1}^{n} a + \sum_{j=1}^{n} jd = na + d \sum_{j=1}^{n} j$$

$$\sum_{j=1}^{n} j = 1 + 2 + 3 + 4 + \ldots + (n-2) + (n-1) + n$$

$$1 + (n-1) = n \quad \text{and} \quad n \quad \ldots \quad n$$

$$\sum_{j=1}^{n} j = n(n+1)/2.$$
Arithmetic series

Example: \[ S = \sum_{j=1}^{5} (2 + j3) = \]
\[ = \sum_{j=1}^{5} 2 + \sum_{j=1}^{5} j3 = \]
\[ = 2\sum_{j=1}^{5} 1 + 3\sum_{j=1}^{5} j = \]
\[ = 2 \times 5 + 3\sum_{j=1}^{5} j = \]
\[ = 10 + 3 \frac{(5+1) \times 5}{2} = \]
\[ = 10 + 45 = 55 \]

Arithmetic series

Example 2: \[ S = \sum_{j=3}^{5} (2 + j3) = \]
\[ = \left[ \sum_{j=1}^{5} (2 + j3) \right] - \left[ \sum_{j=1}^{2} (2 + j3) \right] \quad \text{Trick} \]
\[ = \left[ 2\sum_{j=1}^{5} 1 + 3\sum_{j=1}^{5} j \right] - \left[ 2\sum_{j=1}^{2} 1 + 3\sum_{j=1}^{2} j \right] \]
\[ = 55 - 13 = 42 \]
Double summations

**Example:** $S = \sum_{i=1}^{4} \sum_{j=1}^{2} (2i - j) =$

\[
= \sum_{i=1}^{4} \left[ 2i \sum_{j=1}^{2} - \sum_{j=1}^{2} j \right] =
\]

\[
= \sum_{i=1}^{4} \left[ 2i \cdot 2 - \sum_{j=1}^{2} j \right] =
\]

\[
= \sum_{i=1}^{4} \left[ 2i \cdot 2 - 3 \right] =
\]

\[
= \sum_{i=1}^{4} 4i - 3 =
\]

\[
= 4 \sum_{i=1}^{4} i - 3 \sum_{i=1}^{4} 1 = 4 \cdot 10 - 3 \cdot 4 = 28
\]

Geometric series

**Definition:** The sum of the terms of a geometric progression $a$, $ar$, $ar^2$, ..., $ar^n$ is called a geometric series.

**Theorem:** The sum of the terms of a geometric progression $a$, $ar$, $ar^2$, ..., $ar^n$ is

\[
S = \sum_{j=0}^{n} (ar^j) = a \sum_{j=0}^{n} r^j = a \left[ \frac{r^{n+1} - 1}{r - 1} \right]
\]
Geometric series

**Theorem:** The sum of the terms of a geometric progression $a$, $ar$, $ar^2$, ..., $ar^n$ is

$$S = \sum_{j=0}^{n} (ar^j) = a \sum_{j=0}^{n} r^j = a \left[ \frac{r^{n+1} - 1}{r - 1} \right]$$

**Proof:**

$$S = \sum_{j=0}^{n} ar^j = a + ar + ar^2 + ar^3 + \ldots + ar^n$$

• multiply $S$ by $r$

$$rS = r \sum_{j=0}^{n} ar^j = ar + ar^2 + ar^3 + \ldots + ar^{n+1}$$
Geometric series

**Theorem:** The sum of the terms of a geometric progression \( a, ar, ar^2, \ldots, ar^n \) is

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S = \sum_{j=0}^{n} (ar^j) = a \sum_{j=0}^{n} r^j = a \left[ \frac{r^{n+1} - 1}{r - 1} \right]
\]

**Proof:**

\[
S = \sum_{j=0}^{n} ar^j = a + ar + ar^2 + ar^3 + \ldots + ar^n
\]

• multiply S by \( r \)

\[
rS = r \sum_{j=0}^{n} ar^j = ar + ar^2 + ar^3 + \ldots + ar^{n+1}
\]

• Subtract \( rS - S = [ar + ar^2 + ar^3 + \ldots + ar^{n+1}] - [a + ar + ar^2 + \ldots + ar^n] \)

\[
= ar^{n+1} - a
\]
Geometric series

**Theorem:** The sum of the terms of a geometric progression $a$, $ar$, $ar^2$, ..., $ar^n$ is

$$S = \sum_{j=0}^{n} (ar^j) = a \sum_{j=0}^{n} r^j = a \frac{r^{n+1} - 1}{r - 1}$$

**Proof:**

$$S = \sum_{j=0}^{n} ar^j = a + ar + ar^2 + ar^3 + ... + ar^n$$

- multiply $S$ by $r$

$$rS = r\sum_{j=0}^{n} ar^j = ar + ar^2 + ar^3 + ... + ar^{n+1}$$

- Subtract $rS - S = [ar + ar^2 + ar^3 + ... + ar^{n+1}] - [a + ar + ar^2 + ... + ar^n]$

$$= ar^{n+1} - a$$

$$S = \frac{ar^{n+1} - a}{r - 1} = a \frac{r^{n+1} - 1}{r - 1}$$

**Example:**

$$S = \sum_{j=0}^{3} 2(5)^j =$$

**General formula:**

$$S = \sum_{j=0}^{n} (ar^j) = a \sum_{j=0}^{n} r^j = a \frac{r^{n+1} - 1}{r - 1}$$

$$S = \sum_{j=0}^{3} 2(5)^j = 2 \cdot \frac{5^4 - 1}{5 - 1} =$$

$$= 2 \cdot \frac{625 - 1}{4} = 2 \cdot \frac{624}{4} = 2 \cdot 156 = 312$$
Infinite geometric series

• Infinite geometric series can be computed in the closed form for $x < 1$
• How?

$$\sum_{n=0}^{\infty} x^n = \lim_{k \to \infty} \sum_{n=0}^{k} x^n = \lim_{k \to \infty} \frac{x^{k+1} - 1}{x - 1} = \frac{1}{x - 1}$$

• Thus:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1 - x}$$