CS 441 Discrete Mathematics for CS
Lecture 9

Sets

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Course administration

Homework 3 is out
- **Due on Friday, February 3, 2005**

Midterm 1:
- **Wednesday, February 15, 2006**
- **Covers chapter 1 of the textbook**
- **Closed book**
- **Tables for equivalences and rules of inference will be given to you**

Course web page:

http://www.cs.pitt.edu/~milos/courses/cs441/

Methods of proving theorems

General methods to prove the theorems:

- **Direct proof**
  - $p \to q$ is proved by showing that if p is true then q follows
- **Indirect proof**
  - Show the contrapositive $\neg q \to \neg p$. If $\neg q$ holds then $\neg p$ follows
- **Proof by contradiction**
  - Show that $(p \wedge \neg q)$ contradicts the assumptions
- **Proof by cases**
- **Proofs of equivalence**
  - $p \leftrightarrow q$ is replaced with $(p \to q) \wedge (q \to p)$

Sometimes one method of proof does not go through as nicely as the other method. You may need to try more than one approach.

Proof of equivalences

We want to prove $p \leftrightarrow q$

- Statements: p if and only if q.
- Note that $p \leftrightarrow q$ is equivalent to $[(p \to q) \wedge (q \to p)]$
- Both implications must hold.

Example:

- Integer is odd if and only if n^2 is odd.

Proof of $(p \to q)$:

- **$(p \to q)$** If n is odd then n^2 is odd
- we use a direct proof
- Suppose n is odd. Then n = 2k + 1,where k is an integer.
- n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1
- Therefore, n^2 is odd.

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- Both implications must hold.

- Integer is odd if and only if n^2 is odd.

Proof of $(q \rightarrow p)$:

- $(q \rightarrow p)$: if n^2 is odd then n is odd
- we use an indirect proof $(\neg p \rightarrow \neg q)$ is a contrapositive
- n is even that is n = 2k,
- then n^2 = 4k^2= 2(2k^2)
- Therefore n^2 is even. Done proving the contrapositive.

Since both $(p \rightarrow q)$ and $(q \rightarrow p)$ are true the equivalence is true

Proofs with quantifiers

- **Existence proof**
  - **Constructive**
    - Find the example that shows the statement holds.
  - **Nonconstructive**
    - Show it holds for one example but we do not have the witness example (typically ends with one example or other example)

- **Counterexamples:**
  - use to disprove a universal statements

Sets

Set

- **Definition**: A **set** is a (unordered) collection of objects. These objects are sometimes called **elements** or **members** of the set. (Cantor's naive definition)

- **Examples:**
  - **Vowels in the English alphabet**

    V = { a, e, i, o, u }
  - **First seven prime numbers**.

    X = { 2, 3, 5, 7, 11, 13, 17 }

Representing sets

Representing a set:

1) Listing the members.

2) Definition by property, using set builder notation

{x| x has property P}.

Example:

- Even integers between 50 and 63**.**

1) E = {50, 52, 54, 56, 58, 60, 62}

2) E = {x| 50 <= x < 63, x is an even integer}

If enumeration of the members is hard we often use ellipses.

Example: a set of integers between 1 and 100

- **A= {1,2,3 …, 100}**

Important sets in discrete math

- **Natural numbers:**
  - **N** = {0,1,2,3, …}
- **Integers**
  - **Z** = {…, -2,-1,0,1,2, …}
- **Positive integers**
  - $\mathbf{Z}^+$ = {1,2, 3….}
- **Rational numbers**
  - **Q** = $\{p/q \mid p \in Z, q \in Z, q \neq 0\}$
- **Real numbers**
  - **R**

Russell's paradox

Cantor's naive definition of sets leads to Russell's paradox:

- **Let S = { x | x $\notin$ x },**
  is a set of sets that are not members of themselves.
- **Question:** **Where does the set S belong to?**
  - Is S $\in$ S or S $\notin$ S?
- **Cases**
  - **S $\in$ S ?**

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  - **S $\in$ S ?:** it does not satisfy the condition so it does not hold
  - **S $\notin$ S ?:**

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- **Question: Where does the set S belong to?**
  - Is S $\in$ S or S $\notin$ S?
- **Cases**
  - **S $\in$ S ?:** S does not satisfy the condition so it does not hold that **S $\in$ S**
  - **S $\notin$ S ?:** S is included in the set S and hence S $\notin$ S does not hold
- **A paradox: we cannot decide if S belongs to S or not**
- **Russell's answer: theory of types – used for sets of sets**

Equality

Definition: Two sets are equal if and only if they have the same elements.

Example:

- **{1,2,3} = {3,1,2} = {1,2,1,3,2}**

Note: Duplicates don't contribute anything new to a set, so remove them. The order of the elements in a set doesn't contribute anything new.

Example: Are {1,2,3,4} and {1,2,2,4} equal?

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Example: Are {1,2,3,4} and {1,2,2,4} equal?

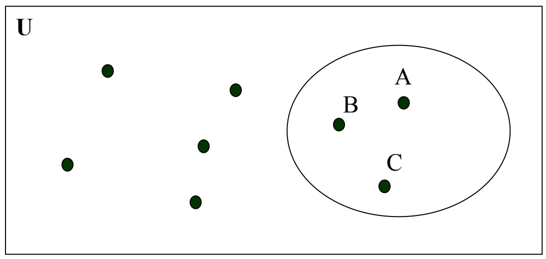
No!

Special sets

- **Special sets:**
  - **The universal set** is denoted by **U:** the set of all objects under the consideration.
  - **The empty set** is denoted as ∅ **or { }.**

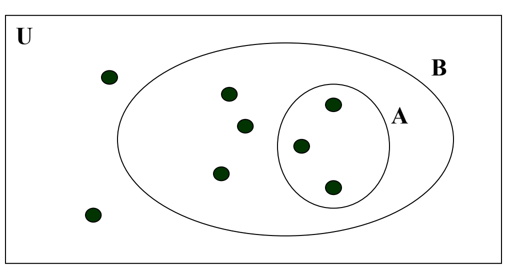
Venn diagrams

- A set can be visualized using **Venn Diagrams**:
  - V={ A, B, C }



A Subset

- **Definition**: A **set** A is said to be a **subset** of B **if and only if** every element of A is also an element of B. We use **A ⊆ B** to indicate **A is a subset of B**.



- Alternate way to define A is a subset of B:

  $\forall x \, (x \in A) \rightarrow (x \in B)$

Empty set/Subset properties

Theorem $\emptyset \subseteq S$

- **Empty set is a subset of any set.**

Proof:

- Recall the definition of a subset: all elements of a set A must be also elements of B: $\forall x\ (x \in A) \rightarrow (x \in B)$.
- We must show the following implication holds for any S
  $\forall x\ (x \in \emptyset) \rightarrow (x \in S)$
- ?

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Proof:

- Recall the definition of a subset: all elements of a set A must be also elements of B: $\forall x\ (x \in A \rightarrow x \in B)$.
- We must show the following implication holds for any S
  $\forall x\ (x \in \emptyset \rightarrow x \in S)$
- Since the empty set does not contain any element, $x \in \emptyset$ is **always False**
- Then the implication is **always True**.

End of proof

Subset properties

Theorem: $S \subseteq S$

- **Any set S is a subset of itself**

Proof:

- the definition of a subset says: all elements of a set A must be also elements of B: $\forall x\ (x \in A) \rightarrow (x \in B)$.
- Applying this to S we get:
- $\forall x\ (x \in S) \rightarrow (x \in S)$ …

Subset properties

Theorem: $S \subseteq S$

- **Any set S is a subset of itself**

Proof:

- the definition of a subset says: all elements of a set A must be also elements of B: $\forall x\ (x \in A \rightarrow x \in B)$.
- Applying this to S we get:
- $\forall x\ (x \in S \rightarrow x \in S)$ which is trivially **True**
- End of proof

Note on equivalence:

- Two sets are equal if each is a subset of the other set.