CS 441 Discrete Mathematics for CS
Lecture 7

Methods of Proof

Milos Hauskrecht
milos@cs.pitt.edu
5329 Sennott Square

Theorems and proofs

- The truth value of some statement about the world is obvious and easy to assign
- The truth of other statements may not be obvious, …

  …. But it may still follow (be derived) from known facts about the world

To show the truth value of such a statement following from other statements we need to provide **a correct supporting argument**

- a proof

Problem:

- It is easy to make a mistake and argue the support incorrectly.

Important questions:

– When is the argument correct?

– How to construct a correct argument, what method to use?

Theorems and proofs

- **Theorem:** a statement that can be shown to be true.
  - **Typically the theorem looks like this:**

    $(p1 \wedge p2 \wedge p3 \wedge \ldots \wedge pn) \rightarrow q$

    Premises (hypotheses) — **conclusion**

- **Example:**

  Fermat's Little theorem:
  - If p is a prime and a is an integer not divisible by p, then: $a^{p-1} \equiv 1 \bmod p$

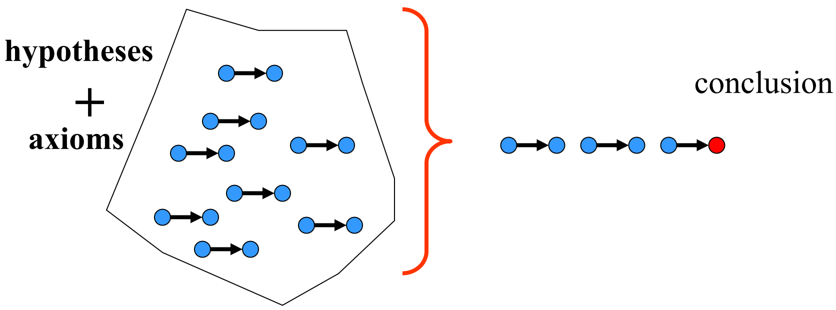
Formal proofs

Proof:
- an argument supporting the validity of the statement
- proof of the theorem:
  - shows that the conclusion follows from premises
  - may use:
    - Premises
    - Axioms
    - Results of other theorems

Formal proofs:
- steps of the proofs follow logically from the set of premises and axioms
- we assume ***formal proofs*** in propositional logic

Formal proofs

- **Formal proofs:**
  - show that steps of the proofs follow logically from the set of hypotheses and axioms



- **In the class we assume formal proofs in propositional logic**

Rules of inference

Rules of inference: **logically valid inference patterns**

Example;

- **Modus Ponens**, or the Law of Detachment
- Rule of inference

$$\begin{array}{l} p \\ \underline{p \rightarrow q} \\ \therefore\ q \end{array}$$

- Given p is true and the implication $p \rightarrow q$ is true then q is true.

Rules of inference

Rules of inference: logically valid inference patterns

Example;

- **Modus Ponens**, or the Law of Detachment
- Rule of inference $\quad p$

  $\underline{p \rightarrow q}$

  $\therefore q$
- Given p is true and the implication $p \rightarrow q$ is true then q is true.

p	q	$p \rightarrow q$
False	*False*	*True*
False	*True*	*True*
True	*False*	*False*
True	*True*	*True*

Rules of inference

Rules of inference: logically valid inference patterns

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Rules of inference

Rules of inference: logically valid inference patterns

Example;

- **Modus Ponens**, or the Law of Detachment
- Rules of inference

$$\begin{array}{l} p \\ \underline{p \rightarrow q} \\ \therefore q \end{array}$$

- Given p is true and the implication $p \rightarrow q$ is true then q is true.
- **Tautology Form:** $(p \wedge (p \rightarrow q)) \rightarrow q$

Rules of inference

- **Addition**

  $p \to (p \lor q)$

  $$\frac{p}{\therefore\ p \lor q}$$

- **Example:** It is below freezing now. Therefore, it is below freezing or raining snow.

- **Simplification**

  $(p \land q) \to p$

  $$\frac{p \land q}{\therefore\ p}$$

- **Example:** It is below freezing and snowing. Therefore it is below freezing.

Rules of inference

- **Modus Tollens**

  $[\neg q \land (p \to q)] \to \neg p$

  $$\begin{array}{l} \neg q \\ \underline{p \to q} \\ \therefore\ \neg p \end{array}$$

- **Hypothetical Syllogism**

  $[(p \to q) \land (q \to r)] \to (p \to r)$

  $$\begin{array}{l} p \to q \\ \underline{q \to r} \\ \therefore\ p \to r \end{array}$$

- **Disjunctive Syllogism**

  $[(p \lor q) \land \neg p] \to q$

  $$\begin{array}{l} p \lor q \\ \underline{\neg p} \\ \therefore\ q \end{array}$$

Rules of inference

- A **valid argument** is one built using the rules of inference from premises (hypotheses). When all premises are true the argument leads to a correct conclusion.
- $(p1 \wedge p2 \wedge p3 \wedge \ldots \wedge pn) \rightarrow q$
- **However, if one or more of the premises is false the conclusion may be incorrect.**
- **How to use the rules of inference?**

Applying rules of inference

Assume the following statements (hypotheses):

- It is not sunny this afternoon and it is colder than yesterday.
- We will go swimming only if it is sunny.
- If we do not go swimming then we will take a canoe trip.
- If we take a canoe trip, then we will be home by sunset.

Show that all these lead to a conclusion:

- We will be home by sunset.

Applying rules of inference

Text:

- It is not sunny this afternoon and it is colder than yesterday.
- We will go swimming only if it is sunny.
- If we do not go swimming then we will take a canoe trip.
- If we take a canoe trip, then we will be home by sunset.

Propositions:

- p = It is sunny this afternoon, q = it is colder than yesterday, r = We will go swimming , s= we will take a canoe trip
- t= We will be home by sunset

Translation:

- **Assumptions:** $\neg p \wedge q$, $r \rightarrow p$, $\neg r \rightarrow s$, $s \rightarrow t$
- **Hypothesis:** t

Applying rules of inference

- **Approach:**
- p = It is sunny this afternoon, q = it is colder than yesterday, r = We will go swimming , s= we will take a canoe trip
- t= We will be home by sunset
- **Translations:**
- **Assumptions:** $\neg p \wedge q$, $r \rightarrow p$, $\neg r \rightarrow s$, $s \rightarrow t$
- **Hypothesis:** t

Translation: "We will go swimming only if it is sunny".

- Ambiguity: $r \rightarrow p$ or $p \rightarrow r$?
- Sunny is a must before we go swimming
- Thus, if we indeed go swimming it must be sunny, therefore $r \rightarrow p$

Proofs using rules of inference

Translations:

- **Assumptions:** $\neg p \wedge q$, $r \rightarrow p$, $\neg r \rightarrow s$, $s \rightarrow t$
- **Hypothesis:** t

Proof:

- 1. $\neg p \wedge q$ Hypothesis
- 2. $\neg p$ Simplification
- 3. $r \rightarrow p$ Hypothesis
- 4. $\neg r$ Modus tollens (step 2 and 3)
- 5. $\neg r \rightarrow s$ Hypothesis
- 6. s Modus ponens (steps 4 and 5)
- 7. $s \rightarrow t$ Hypothesis
- 8. t Modus ponens (steps 6 and 7)
- **end of proof**