CS 441 Discrete Mathematics for CS
Lecture 4

Predicate logic

Milos Hauskrecht
milos@cs.pitt.edu
5329 Sennott Square

Course administration

Homework 1

- **Is due on Wednesday, January 25, 2006.**

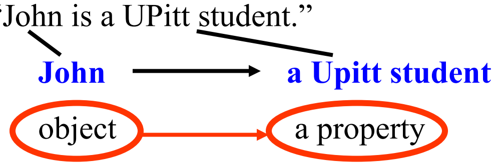
- **Recitations for Homework 1:**
  - **Today, Wednesday, January 18, 2005**
  - **Monday, January 23, 2006**

Course web page:

http://www.cs.pitt.edu/~milos/courses/cs441/

Limitations of the propositional logic

- **Propositional logic:** the world is described in terms of propositions
- **A proposition is** a statement that is either true or false.
- **Limitations:**
  - **objects** in elementary statements, their properties and relations are not explicitly represented in the propositional logic
- **Example:**
  - "John is a UPitt student."



  - Objects and properties are hidden in the statement, it is not possible to reason about them

Limitations of the propositional logic

- **Statements for groups of objects**
  - In propositional logic these must be exhaustively enumerated
- **Example:**
  - If John is a CS UPitt graduate then John has passed cs441

  Translation:
  - John is a CS UPitt graduate → John has passed cs441

  Similar statements can be written for other Upitt graduates:
  - Ann is a CS Upitt graduate → Ann has passed cs441
  - Ken is a CS Upitt graduate → Ken has passed cs441
  - …
- **What is a more natural solution to express the above knowledge?**

Limitations of the propositional logic

- **Statements for groups of objects**
  - In propositional logic these must be exhaustively enumerated
- **Example:**
  - If John is a CS UPitt graduate then John has passed cs441

  Translation:
  - John is a CS UPitt graduate → John has passed cs441

  Similar statements can be written for other Upitt graduates:
  - Ann is a CS Upitt graduate → Ann has passed cs441
  - Ken is a CS Upitt graduate → Ken has passed cs441
  - …
- **Solution:** make statements with **variables**
  - If x is a CS Upitt graduate then x has passed cs441
  - x is a CS UPitt graduate → x has passed cs441

Predicate logic

Remedies the limitations of propositional logic
- Explicitly models objects and their properties
- Allows to make statements with variables and quantify them

Basic building blocks of the predicate logic:
- **Constant** –models a specific object

  Examples: "John", "France", "7"
- **Variable** – represents object of specific type **(defined by the universe of discourse)**

  Examples: x, y

  (universe of discourse can be people, students, numbers)
- **Predicate -** over one, two or many variables or constants.
  - Represents properties or relations among objects

  Examples: Red(car23), student(x), married(John,Ann)

Predicates

Predicates represent properties or relations among objects

A predicate P(x) assigns a value **true or false** to each x depending on whether the property holds or not for x.

- The assignment is best viewed as a big table with the variable x substituted for objects from the universe of discourse

Example:

- Assume **Student(x)** where the universe of discourse are people
- Student(John) …. T (if John is a student)
- Student(Ann) …. T (if Ann is indeed a student)
- Student(Jane) ….. F (if Jane is not a student)
- …

Predicates

Assume a predicate P(x) that represents the statement:

- **x is a prime number**

  Note: A positive integer is a prime if it is divisible only by 1 and itself

What are the truth values of:

- P(2) ?

Predicates

Assume a predicate P(x) that represents the statement:

- **x is a prime number**

What are the truth values of:

- P(2) T
- P(3) ?

Predicates

Assume a predicate P(x) that represents the statement:

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What are the truth values of:

- P(2) T
- P(3) T
- P(4) ?

Predicates

Assume a predicate P(x) that represents the statement:

- **x is a prime number**

What are the truth values of:

- P(2) T
- P(3) T
- P(4) F
- P(5) ?

Predicates

Assume a predicate P(x) that represents the statement:

- **x is a prime number**

What are the truth values of:

- P(2) T
- P(3) T
- P(4) F
- P(5) T
- P(6) ?

Predicates

Assume a predicate P(x) that represents the statement:

- **x is a prime number**

What are the truth values of:

- P(2) T
- P(3) T
- P(4) F
- P(5) T
- P(6) F
- P(7) ?

Predicates

Assume a predicate P(x) that represents the statement:

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What are the truth values of:

- P(2) T
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- P(4) F
- P(5) T
- P(6) F
- P(7) T

Predicates

Assume a predicate P(x) that represents the statement:

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What are the truth values of:

- P(2) T
- P(3) T
- P(4) F
- P(5) T
- P(6) F
- P(7) T

All statements P(2), P(3), P(4), P(5), P(6), P(7) are propositions

Predicates

Assume a predicate P(x) that represents the statement:

- **x is a prime number**

What are the truth values of:

- P(2) T
- P(3) T
- P(4) F
- P(5) T
- P(6) F
- P(7) T

Is P(x) a proposition?

Predicates

Assume a predicate P(x) that represents the statement:

- **x is a prime number**

What are the truth values of:

- P(2) T
- P(3) T
- P(4) F
- P(5) T
- P(6) F
- P(7) T

Is P(x) a proposition? No. Many possible substitutions are possible.

Predicates

Important:

- predicate P(x) is **not a proposition** since there are more objects it can be applied to

This is the same as in propositional logic …

… But the difference is:

- propositional logic does not let us go inside the statements and manipulate x
- predicate logic allows us to explicitly manipulate and substitute objects

Predicates

- Predicates can have **more arguments** which represent the **relations between objects**

Example:

- Older(John, Peter) denotes 'John is older than Peter'
  - this is a proposition because it is either true or false
- Older(x,y) - 'x is older than y'
  - not a proposition, but after the substitution it becomes one

Predicates

- Predicates can have **more arguments** which represent the **relations between objects**

Example:

- Let Q(x,y) denote 'x+5 >y'
  - Is Q(x,y) a proposition?

Predicates

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Example:

- Let Q(x,y) denote ‘x+5 >y’
  - Is Q(x,y) a proposition? **No!**
  - Is Q(3,7) a proposition?

Predicates

- Predicates can have **more arguments** which represent the **relations between objects**

Example:

- Let Q(x,y) denote ‘x+5 >y’
  - Is Q(x,y) a proposition? **No!**
  - Is Q(3,7) a proposition? **Yes.** It is true.

Predicates

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Example:

- Let Q(x,y) denote 'x+5 >y'
  - Is Q(x,y) a proposition? **No!**
  - Is Q(3,7) a proposition? **Yes.** Its truth value is true.
  - What is the truth value of:
  - Q(1,6) **?**

Predicates

- Predicates can have **more arguments** which represent the **relations between objects**

Example:

- Let Q(x,y) denote 'x+5 >y'
  - Is Q(x,y) a proposition? **No!**
  - Is Q(3,7) a proposition? **Yes.** It is true.
  - What is the truth value of:
  - Q(1,6) F
  - Q(2,2) ?

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  - Is Q(x,y) a proposition? **No!**
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  - What is the truth value of
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  - Q(1,6) F
  - Q(2,2) T
  - Is Q(3,y) a proposition?

Predicates

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Example:

- Let Q(x,y) denote 'x+5 >y'
  - Is Q(x,y) a proposition? **No!**
  - Is Q(3,7) a proposition? **Yes.** It is true.
  - What is the truth value of
  - Q(3,7) T
  - Q(1,6) F
  - Q(2,2) T
  - Is Q(3,y) a proposition? **No!** We cannot say if it is true or false.

Quantified statements

Predicate logic allows us to make statements about groups of objects

- To do this we use special quantified expressions

Two types of quantified statements:

- **universal**

  Example: ' all CS Upitt graduates have to pass cs441"
  - the statement is true for all graduates
- **existential**

  Example: 'Some CS Upitt students graduate with honor.'
  - the statement is true for some people

Universal quantifier

Defn: The universal quantification of P(x) is the proposition "P(x) is true for all values of x in the universe of discourse." The notation **∀x P(x)** denotes the universal quantification of P(x), and is expressed as **for every x, P(x).**

Example:

- Let P(x) denote x > x - 1.
- What is the truth value of ∀x P(x)?
- Assume the universe of discourse of x is all real numbers.

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Example:

- Let P(x) denote x > x - 1.
- What is the truth value of ∀x P(x)?
- Assume the universe of discourse of x is all real numbers.
- **Answer:** Since every number x is greater than itself minus 1. Therefore, **∀x P(x) is true.**

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Example 2:

- Let T(x) denote x > 5.
- What is the truth value of ∀x T(x)?
- Assume the universe of discourse of x are real numbers
- **Answer:** ?

Universal quantifier

Definition: The **universal quantification** of P(x) is the proposition "P(x) is true for all values of x in the universe of discourse." The notation **∀x P(x)** denotes the universal quantification of P(x), and is expressed as **for every x, P(x).**

Example 2:

- Let T(x) denote x > 5.
- What is the truth value of ∀x T(x)?
- Assume the universe of discourse of x is all real numbers.
- **Answer:**
  - Since 3 > 5 is false. So, T(x) is not true for all values of x. Therefore, it is **false that ∀x T(x).**

Universal quantifier

Quantification converts a propositional function (e.g. P(x)) **into a proposition** by binding a variable to a set of values from the universe of discourse.

Example:

- Let P(x) denote x > x - 1.
- Is P(x) a proposition?

Universal quantifier

Quantification converts a propositional function into **a proposition** by binding a variable to a set of values from the universe of discourse.

Example:

- Let P(x) denote x > x - 1.
- Is P(x) a proposition? **No.** Many possible substitutions.
- Is $\forall$x P(x) a proposition?

Universal quantifier

Quantification converts a propositional function into **a proposition** by binding a variable to a set of values from the universe of discourse.

Example:

- Let P(x) denote x > x - 1.
- Is P(x) a proposition? **No.** Many possible substitutions.
- Is $\forall$x P(x) a proposition? **Yes.** True if for all x from the universe of discourse P(x) is true.
- Is $\forall$x Q(x,y) a proposition?

Universal quantifier

Quantification converts a propositional function into **a proposition** by binding a variable to a set of values from the universe of discourse.

Example:

- Let P(x) denote x > x - 1.
- Is P(x) a proposition? **No.** Many possible substitutions.
- Is $\forall$x P(x) a proposition? **Yes.** True if for all x from the universe of discourse P(x) is true. Which holds?
- Is $\forall$x Q(x,y) a proposition? **No.** The variable y is free and can be substituted by many objects.

Existential quantifier

Definition: The **existential quantification** of P(x) is the proposition "There exists an element in the universe of discourse such that P(x) is true." The notation **∃x P(x)** denotes the existential quantification of P(x), and is expressed as **there is an x such that P(x) is true**.

Example 1:

- Let T(x) denote x > 5 and x is from real numbers
- What is the truth value of ∃x T(x)?
- **Answer:** ?

Existential quantifier

Definition: The **existential quantification** of P(x) is the proposition "There exists an element in the universe of discourse such that P(x) is true." The notation **∃x P(x)** denotes the existential quantification of P(x), and is expressed as **there is an x such that P(x) is true**.

Example 1:

- Let T(x) denote x > 5 and x is from Real numbers.
- What is the truth value of ∃x T(x)?
- **Answer:**
- Since 10 > 5 is true. Therefore, it is **true that ∃x T(x).**

Existential quantifier

Definition: The **existential quantification** of P(x) is the proposition "There exists an element in the universe of discourse such that P(x) is true." The notation **$\exists$x P(x)** denotes the existential quantification of P(x), and is expressed as **there is an x such that P(x) is true**.

Example 2:

- Let Q(x) denote x = x + 2 where x is real numbers
- What is the truth value of $\exists$x Q(x)?
- **Answer:** ?

Existential quantifier

Definition: The **existential quantification** of P(x) is the proposition "There exists an element in the universe of discourse such that P(x) is true." The notation **$\exists$x P(x)** denotes the existential quantification of P(x), and is expressed as **there is an x such that P(x) is true**.

Example 2:

- Let Q(x) denote x = x + 2 where x is real numbers
- What is the truth value of $\exists$x Q(x)?
- **Answer:** Since no real number is 2 larger than itself, the truth value of **$\exists$x Q(x) is false**.