CS 441 Discrete Mathematics for CS
Lecture 38

Relations

Milos Hauskrecht
milos@cs.pitt.edu
5329 Sennott Square

Composite of relations

Definition: Let R be a relation on a set A. The **powers R^n**, n = 1,2,3,... is defined inductively by

- $R^1 = R$ and $R^{n+1} = R^n \circ R$.

Examples

- R = {(1,2),(2,3),(2,4), (3,3)} is a relation on A = {1,2,3,4}.
- $R^1 = R$ = {(1,2),(2,3),(2,4), (3,3)}
- R^2 = {(1,3), (1,4), (2,3), (3,3)}
- R^3 = {(1,3), (2,3), (3,3)}
- R^4 = {(1,3), (2,3), (3,3)}
- $R^k = R^3$, $k > 3$.

Transitive relation

Definition (transitive relation): A relation R on a set A is called **transitive** if

- $[(a,b) \in R \text{ and } (b,c) \in R] \rightarrow (a,c) \in R$ for all $a, b, c \in A$.

- **Example 1:**
- $R_{div} = \{(a\ b), \text{ if } a \mid b\}$ on $A = \{1,2,3,4\}$
- $R_{div} = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$
- **Is R_{div} transitive?**
- **Answer: Yes.**

Connection to R^n

Theorem: The relation R on a set A is transitive if and only if $R^n \subseteq R$ for $n = 1,2,3,\ldots$.

Proof: biconditional (if and only if)

(←) Suppose $R^n \subseteq R$, for $n = 1,2,3,\ldots$.

- Let $(a,b) \in R$ and $(b,c) \in R$
- by the definition of $R \circ R$, $(a,c) \in R \circ R \subseteq R \rightarrow$
- R is transitive.

Connection to R^n

Theorem: The relation R on a set A is transitive if and only if $R^n \subseteq R$ for n = 1,2,3,... .

Proof: biconditional (if and only if)

(→) Suppose R is transitive. Show $R^n \subseteq R$, for n =1,2,3,... .

- Let P(n) : $R^n \subseteq R$. Math induction.
- **Basis Step:** P(1) says $R^1 = R$ so, $R^1 \subseteq R$ is true.
- **Inductive Step:** show $P(n) \rightarrow P(n+1)$
- Want to show if $R^n \subseteq R$ then $R^{n+1} \subseteq R$.
- Let $(a,b) \in R^{n+1}$ then by the definition of $R^{n+1} = R^n \circ R$ there is an element $x \in A$ so that $(a,x) \in R$ and $(x,b) \in R^n \subseteq R$ (inductive hypothesis). In addition to $(a,x) \in R$ and $(x,b) \in R$, R is transitive; so $(a,b) \in R$.
- Therefore, $R^{n+1} \subseteq R$.

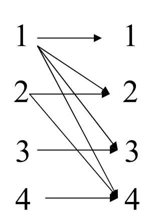
Representing binary relations with graphs

- We can graphically represent a binary relation R from A to B as follows:
  - if **a R b** then draw an arrow from a to b.

a → b

Example:

- Relation R_{div} (from previous lectures) on A={1,2,3,4}
- R_{div} = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)}

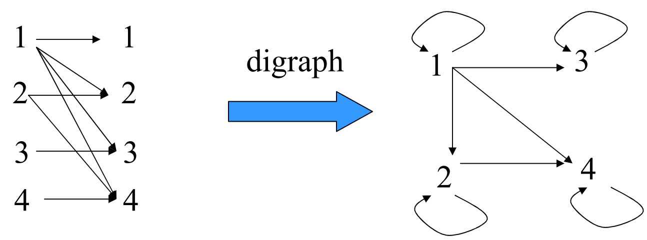


Representing relations on a set with digraphs

Definition: A **directed graph or digraph** consists of a set of vertices (or nodes) together with a set E of ordered pairs of elements of V valled edges (or arcs). The vertex a is called the initial vertex of the edge (a,b) and vertex b is the terminal vertex of this edge. An edge of the form (a,a) is called a loop.

Example

- Relation R_{div} ={(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)}



Closures of relations

- Let R={(1,1),(1,2),(2,1),(3,2)} on A ={1 2 3}.
- Is this relation reflexive?
- Answer: ?

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- Is this relation reflexive?
- Answer: **No.** Why?
- **(2,2) and (3,3) is not in R.**

- The question is what is **the minimal relation S** $\supseteq$ R that is reflexive?
- How to make R reflexive with minimum number of additions?
- Answer: ?

Closures of relations

- Let R={(1,1),(1,2),(2,1),(3,2)} on A ={1 2 3}.
- Is this relation reflexive?
- Answer: **No.** Why?
- **(2,2) and (3,3) is not in R.**

- The question is what is **the minimal relation S** $\supseteq$ R that is reflexive?
- How to make R reflexive with minimum number of additions?
- **Answer:** Add (2,2) and (3,3)
  - Then S= {(1,1),(1,2),(2,1),(3,2),(2,2),(3,3)}
  - R $\subseteq$ S
  - The minimal set S $\supseteq$ R is called **the reflexive closure of R**

Reflexive closure

The set S is called **the reflexive closure** of R if it:

- contains R
- has reflexive property
- is contained in every reflexive relation Q that contains R (R $\subseteq$ Q) , that is S $\subseteq$ Q

Closures on relations

- **Relations can have different properties:**
  - **reflexive,**
  - **symmetric**
  - **transitive**

- Because of that we can have:
  - **symmetric,**
  - **reflexive and**
  - **transitive**

  closure.

Closures

Definition: Let R be a relation on a set A. A relation S on A with property P is called **the closure of R with respect to P** if S is a subset of every relation Q ($S \subseteq Q$) with property P that contains R ($R \subseteq Q$).

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Example (symmetric closure):

- Assume R={(1,2),(1,3), (2,2)} on A={1,2,3}.
- What is the symmetric closure S of R?
- S=?

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Example (symmetric closure):

- Assume R={(1,2),(1,3), (2,2)} on A={1,2,3}.
- What is the symmetric closure S of R?
- S = {(1,2),(1,3), (2,2)} $\cup$ {(2,1), (3,1)}
  = {(1,2),(1,3), (2,2),(2,1), (3,1)}

Closures

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- **Example (transitive closure):**
- Assume R={(1,2), (2,2), (2,3)} on A={1,2,3}.
- **Is R transitive?**

Closures

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- **Example (transitive closure):**
- Assume R={(1,2), (2,2), (2,3)} on A={1,2,3}.
- **Is R transitive? No.**
- **How to make it transitive?**
- **S = ?**

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- **Example (transitive closure):**
- Assume R={(1,2), (2,2), (2,3)} on A={1,2,3}.
- **Is R transitive? No.**
- **How to make it transitive?**
- **S = {(1,2), (2,2), (2,3)} ∪ {(1,3)}**
  = {(1,2), (2,2), (2,3),(1,3)}
- S is the transitive closure of R

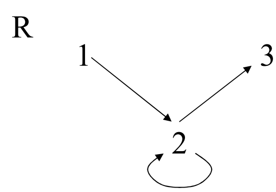
Transitive closure

We can represent the relation on the graph. Finding a transitive closure corresponds to finding all pairs of elements that are connected with a directed path (or digraph).

Example:

Assume R={(1,2), (2,2), (2,3)} on A={1,2,3}.

Transitive closure S = {(1,2), (2,2), (2,3),(1,3)}.



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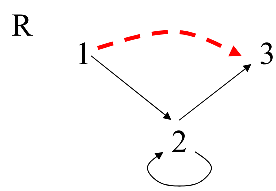

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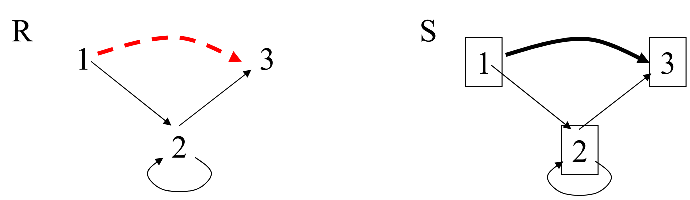
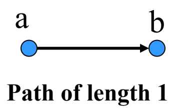
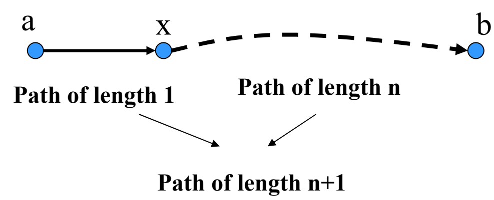


Transitive closure

Theorem: Let R be a relation on a set A. There is a path of length n from a to b if and only if $(a,b) \in R^n$.

Proof (math induction):



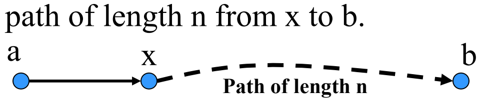


Transitive closure

Theorem: Let R be a relation on a set A. There is a path of length n from a to b if and only if $(a,b) \in R^n$.

Proof (math induction):

- **P(1):** There is a path of length 1 from a to b if and only if $(a,b) \in R^1$, by the definition of R.
- **Show $P(n) \rightarrow P(n+1)$:** Assume there is a path of length n from a to b if and only if $(a,b) \in R^n \rightarrow$ there is a path of length n+1 from a to b if and only if $(a,b) \in R^{n+1}$.
- There is a path of length n+1 from a to b if and only if there exists an $x \in A$, such that $(a,x) \in R$ (a path of length 1) and $(x,b) \in R^n$ is a path of length n from x to b.



- $(x,b) \in R^n$ holds due to P(n). Therefore, there is a path of length n + 1 from a to b. This also implies that $(a,b) \in R^{n+1}$.

Connectivity relation

Definition: Let R be a relation on a set A. The **connectivity relation** R* consists of all pairs (a,b) such that there is a path (of any length, i.e. 1 or 2 or 3 or ...) between a and b in R.

$$R^* = \bigcup_{k=1}^{\infty} R^k$$

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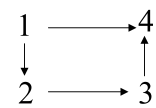

- A = {1,2,3,4}
- R = {(1,2),(1,4),(2,3),(3,4)}

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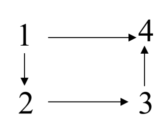

- A = {1,2,3,4}
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- R^2 = ?
-

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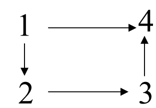

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- R^3 = ?
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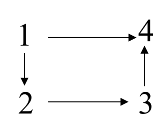

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- R^3 = {(1,4)}
- R4 = ?

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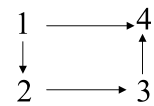

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- ...
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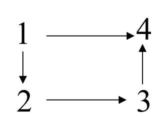
Example:



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- R^3 = {(1,4)}
- R4 = ∅
- ...
- R* = {(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)}

Transitivity closure and connectivity relation

Theorem: The transitive closure of a relation R **equals** the connectivity relation R*.

Based on the following **Lemma.**

Lemma 1: Let A be a set with n elements, and R a relation on A. If there is a path from a to b, then there exists a path of length $\leq$ n in between (a,b). Consequently:

$$R^* = \bigcup_{k=1}^{n} R^k$$

Connectivity

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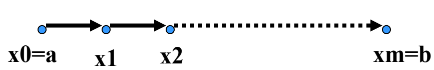
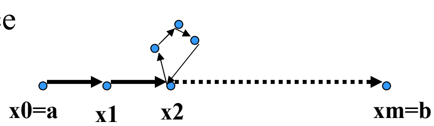
Proof (intuition):

- There are at most n different elements we can visit on a path if the path does not have loops



- Loops may increase the length but the same node is visited more than once



Connectivity

Lemma 1: Let A be a set with n elements, and R a relation on A. If there is a path from a to b, then there exists a path of length < n in between (a,b). Consequently:

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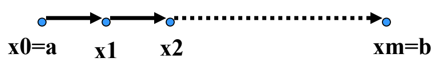

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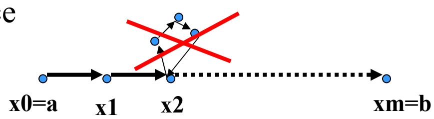

Equivalence relation

Definition: A relation R on a set A is called an **equivalence relation** if it is reflexive, symmetric and transitive.

Example: Let A = {0,1,2,3,4,5,6} and

- R= {(a,b)| a,b $\in$ A, a $\equiv$ b mod 3} (a is congruent to b modulo 3)

Congruencies:

- 0 mod 3 = ?

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- 0 mod 3 = 0   1 mod 3 = ?

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- 0 mod 3 = 0   1 mod 3 = 1   2 mod 3 = 2   3 mod 3 = ?

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- 0 mod 3 = 0   1 mod 3 = 1   2 mod 3 = 2   3 mod 3 = 0
- 4 mod 3 = ?

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- 4 mod 3 = 1   5 mod 3 = 2   6 mod 3 = 0

Relation R has the following pairs:

?

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Congruencies:

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Relation R has the following pairs:

- (0,0) (0,3), (3,0), (0,6), (6,0)
- (3,3), (3,6) (6,3), (6,6) (1,1),(1,4), (4,1), (4,4)
- (2,2), (2,5), (5,2), (5,5)

Equivalence relation

- **Relation R on A={0,1,2,3,4,5,6} has the following pairs:**
  (0,0) (0,3), (3,0), (0,6), (6,0)
  (3,3), (3,6) (6,3), (6,6) (1,1),(1,4), (4,1), (4,4)
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- Is R reflexive?

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- Is R reflexive? **Yes.**
- Is R symmetric?

Equivalence relation

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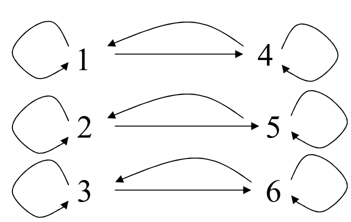
- Is R reflexive? **Yes.**
- Is R symmetric? **Yes.**
- Is R transitive?



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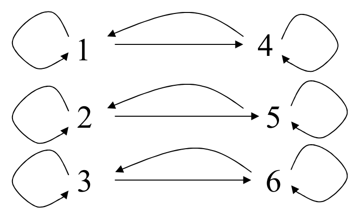
- Is R reflexive? **Yes.**
- Is R symmetric? **Yes.**
- Is R transitive. **Yes.**



Then

- **R is an equivalence relation**.

Equivalence class

Definition: Let R be an equivalence relation on a set A. The set $\{ x \in A \mid a R x\}$ is called **the equivalence class of a,** denoted by $[a]_R$ or simply [a] when there is only one relation R. If $b \in [a]$ then b is called **a representative of this equivalence class**.

Example:

- Assume $R=\{(a,b) \mid a \equiv b \bmod 3\}$ for A={0,1,2,3,4,5,6}
- Pick an element a =0.
- $[0]_R = \{0,3,6\}$
- Element 1: $[1]_R = ?$

Equivalence class

Definition: Let R be an equivalence relation on a set A. The set { x ∈ A | a R x} is called **the equivalence class of a,** denoted by $[a]_R$ or simply [a] when there is only one relation R. If b ∈ [a] then b is called **a representative of this equivalence class**.

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- Element 3: $[3]_R$= ?

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- Element 3: $[3]_R$= {0,3,6} = $[0]_R$
- Element 4: $[4]_R$= ?

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- Element 3: $[3]_R = \{0,3,6\} = [0]_R$
- Element 4: $[4]_R = \{1,4\} = [1]_R$
- Element 5: $[5]_R = ?$

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- Element 5: $[5]_R = \{2,5\} = [2]_R$

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- $[0]_R = \{0,3,6\}$
- Element 1: $[1]_R= \{1,4\}$
- Element 2: $[2]_R= \{2,5\}$
- Element 3: $[3]_R= \{0,3,6\} = [0]_R = [6]_R$
- Element 4: $[4]_R= \{1,4\} = [1]_R$
- Element 5: $[5]_R= \{2,5\} = [2]_R$

Equivalence class

Example:

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Three different equivalence classes all together:

- $[0]_R = [3]_R =[6]_R = \{0,3,6\}$
- $[1]_R= [4]_R= \{1,4\}$
- $[2]_R= [5]_R= \{2,5\}$