CS 441 Discrete Mathematics for CS
Lecture 34

Relations

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Course administration

- **Homework assignment 11**
  - **due on Friday, April 21, 2006**
- **Final exam**
  - **Thursday, April 27, 2006**
  - **At 12:00-1:50pm**
  - **The same room as lectures**

Course web page:

http://www.cs.pitt.edu/~milos/courses/cs441/

Cartesian product (review)

- Let $A=\{a_1, a_2, ..a_k\}$ and $B=\{b_1,b_2,..b_m\}$.
- **The Cartesian product** A x B is defined by a set of pairs $\{(a_1\ b_1), (a_1, b_2), \ldots (a_1, b_m), \ldots, (a_k,b_m)\}$.

Example:

Let A={a,b,c} and B={1 2 3}. What is AxB?

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Example:

Let A={a,b,c} and B={1 2 3}. What is AxB?

AxB = {(a,1),(a,2),(a,3),(b,1),(b,2),(b,3)}

Binary relation

Definition: Let A and B be sets. A **binary relation from A to B** is **a subset of a Cartesian product A x B**.

- Let R ⊆ A x B means R is a set of ordered pairs of the form (a,b) where a ∈ A and b ∈ B.
- We use the notation **a R b to denote (a,b) ∈ R** and **a R̸ b to denote (a,b) ∉ R**. If **a R b**, we say a is related to b by R.

Example: Let A={a,b,c} and B={1,2,3}.

- Is R={(a,1),(b,2),(c,2)} a relation from A to B?

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- Is P={(a,a),(b,c),(b,a)} a relation from A to A?

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- Is P={(a,a),(b,c),(b,a)} a relation from A to A? **Yes**

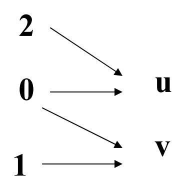
Representing binary relations

- We can graphically represent a binary relation R as follows:
  - if **a R b** then draw an arrow from a to b.

a → b

Example:

- Let A = {0, 1, 2}, B = {u,v} and R = { (0,u), (0,v), (1,v), (2,u) }
- Note: R ⊆ A x B.
- **Graph:**



Representing binary relations

- We can represent a binary relation R by a **table** showing (marking) the ordered pairs of R.

Example:

- Let A = {0, 1, 2}, B = {u,v} and R = { (0,u), (0,v), (1,v), (2,u) }
- **Table:**

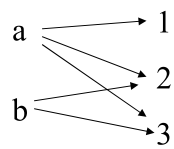
R	u	v
0	x	x
1		x
2	x	

or

R	u	v
0	1	1
1	0	1
2	1	0

Relations and functions

- Relations represent **one to many relationships** between elements in A and B.
- **Example:**



- What is the difference between a **relation and a function from A to B**?

Relations and functions

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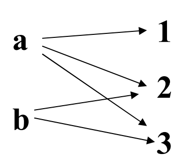
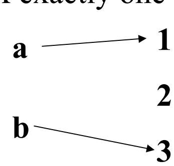


- What is the difference between a **relation and a function from A to B**? A function on sets A,B A → B assigns to each element in the domain set A exactly one element from B. So it is **a special relation.**



Relation on the set

Definition: **A relation on the set A** is a relation from A to itself.

Example 1:

- Let A = {1,2,3,4} and R_{div} = {(a,b)| a divides b}
- What does R_{div} consist of?
- R_{div} = { …

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Example 1:

- Let A = {1,2,3,4} and R_{div} = {(a,b)| a divides b}
- What does R_{div} consist of?
- R_{div} = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)}
-

R	1	2	3	4
1	x	x	x	x
2		x		x
3			x	
4				x

Relation on the set

Example:

- Let A = {1,2,3,4}.
- Define a $R_{\neq}$ b if and only if a $\neq$ b.

$R_{\neq}$={ …

Relation on the set

Example:

- Let A = {1,2,3,4}.
- Define a $R_{\neq}$ b if and only if a $\neq$ b.

$R_{\neq}$={(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,4),(4,1),(4,2),(4,3)}

-

R	1	2	3	4
1		x	x	x
2	x		x	x
3	x	x		x
4	x	x	x	

Binary relations

- **Theorem:** The number of binary relations on a set A, where | A | = n is:

$$2^{n^2}$$

- **Proof:**
- If | A | = n then the cardinality of the Cartesian product | A x A | = n^2.
- R is a binary relation on A if R $\subseteq$ A x A (that is, R is a subset of A x A).
- The number of subsets of A x A is : $2^{|AxA|} = 2^{n^2}$

Binary relations

- **Example**: Let A = {1,2}
- What is A x A = **{(1,1),(1,2),(2,1),(2,2)}**
- **List of possible relations (subsets of A x A):**
- $\varnothing$ …. 1
- {(1,1)} {(1,2)} {(2,1)} {(2,2)} …. 4
- {(1,1), (1,2)} {(1,1),(2,1)} {(1,1),(2,2)} …. 6
  {(1,2),(2,1)} {(1,2),(2,2)} {(2,1),(2,2)}
- {(1,1),(1,2),(2,1)} {(1,1),(1,2),(2,2)} …. 4
  {(1,1),(2,1),(2,2)} {(1,2),(2,1),(2,2)}
- {(1,1),(1,2),(2,1),(2,2)} …. 1

(total: **16**)

- Use formula: $\mathbf{2^4 = 16}$