CS 441 Discrete Mathematics for CS
Lecture 33

Probabilities

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Course administration

- **Homework assignment 10 is out**
  - **due on Friday, April 14, 2006**
- **Final exam (confirmation is still pending)**
  - **Thursday, April 27, 2006**
  - **At 12:00-1:50pm**
  - **The same room as lectures**

Course web page:

http://www.cs.pitt.edu/~milos/courses/cs441/

Random variable

Sample space: a set of all outcomes of the experiment

- **Example:** roll of two dices

  {(1,1),(1,2),(1,3), …(1,6),(2,1),……(6,5),(6,6)}

Definition: A random variable X is a function from the sample space S of an experiment to the set of real numbers **X: S → R.** A random variable assigns a number to each possible outcome.

- **Example:** a random variable that describes the sum of two dices

  (1,1) → 2, (1,2)→3, (2,1)→3, … (6,5)→11, (6,6)→12

Probability distribution

The distribution of a random variable X on the sample space S is a set of pairs (r p(X=r)) for all r in S where r is the number and p(X=r) is the probability that X takes a value r.

Example: the random variable is the sum of two dices

(1,1) → 2, (1,2)→3, (2,1)→3, … (6,5)→11, (6,6)→12

Distribution of the random variable:

(2 1/36)

(3 2/36)

(7 1/6)

…

(12 1/36)

Bernoulli distribution

- **Coin flip**
- P(head) =0.6 and the probability of a tail is 0.4. Assume 5 coin flips such that each coin flip is **independent** of the previous one
- What is the probability of seeing:
  - HHHHH - 5 heads in a row
- **P(HHHHH) = ?**

Probabilities

- **Repeated coin flip**
- P(head) =0.6 and the probability of a tail is 0.4. Each coin flip is **independent** of the previous one
- What is the probability of seeing:
  - HHHHH - 5 heads in a row
- **P(HHHHH) = 0.6^5 =**
  - Assume the outcome is HHTTT
- **P(HHTTT)= ?**

Probabilities

- **Repeated coin flip**
- P(head) =0.6 and the probability of a tail is 0.4. Each coin flip is **independent** of the previous one
- What is the probability of seeing:
  - HHHHH - 5 heads in a row
- **P(HHHHH) = 0.6^5 =**
  - Assume the outcome is HHTTT
- **P(HHTTT)= $0.6*0.6*0.4^3 = 0.6^2*0.4^3$**
  - Assume the outcome is TTHHT
- **P(TTHHT)=?**

Probabilities

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- P(head) =0.6 and the probability of a tail is 0.4. Each coin flip is **independent** of the previous one
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- **P(TTHHT)=$0.4^2*0.6^2*0.4 = 0.6^2*0.4^3$**

Probabilities

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- **P(TTHHT)=$0.4^2*0.6^2*0.4 = 0.6^2*0.4^3$**
- What is the probability of seeing three tails and two heads?

Probabilities

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- What is the probability of seeing:
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- **P(HHHHH) = 0.6^5 =**
  - Assume the outcome is HHTTT
- **P(HHTTT)= $0.6*0.6*0.4^3 = 0.6^2*0.4^3$**
  - Assume the outcome is TTHHT
- **P(TTHHT)=$0.4^2*0.6^2*0.4 = 0.6^2*0.4^3$**
- What is the probability of seeing three tails and two heads?
- The number of two-head-three tail combinations?

Probabilities

- **Repeated coin flip**
- P(head) =0.6 and the probability of a tail is 0.4. Each coin flip is **independent** of the previous one
- What is the probability of seeing:
  - HHHHH - 5 heads in a row
- **P(HHHHH) = 0.6^5 =**
  - Assume the outcome is HHTTT
- **P(HHTTT)= $0.6*0.6*0.4^{3}$= $0.6^2*0.4^3$**
  - Assume the outcome is TTHHT
- **P(TTHHT)=$0.4^{2}*0.6^{2}*0.4 = 0.6^{2}*0.4^{3}$**
- What is the probability of seeing three tails and two heads?
- The number of two-head-three tail combinations = **C(2,5)**
- P(two-heads-three tails) = **C(2,5) *0.6^2 *0.4^3**

Probabilities

- **Repeated coin flip problem**
- Assume **the random variable is the count of occurrences of heads** in 5 coin flips. For example:
- TTTTT yields outcome 0
- HTTTT or TTHTT yields 1
- HTHHT yields 3 …

- What is the probability of an outcome 0?
- P(outcome=0) = **0.6^0 *0.4^5**
- P(outcome=1) =?

Probabilities

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- TTTTT yields outcome 0
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- What is the probability of an outcome 0?
- P(outcome=0) = **C(5,0) $0.6^0 * 0.4^5$**
- P(outcome=1) = **?**

Probabilities

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- TTTTT yields outcome 0
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- What is the probability of an outcome 0?
- P(outcome=0) = **C(5,0) $0.6^0 * 0.4^5$**
- P(outcome=1) = **C(5,1) $0.6^1 * 0.4^4$**
- P(outcome =2) = **?**

Probabilities

- **Repeated coin flip problem**
- Assume **the random variable is the count of occurrences of heads** in 5 coin flips. For example:
- TTTTT yields outcome 0
- HTTTT or TTHTT yields 1
- HTHHT yields 3 …

- What is the probability of an outcome 0?
- P(outcome=0) = **C(5,0) 0.6^0 *0.4^5**
- P(outcome=1) = **C(5,1) 0.6^1 *0.4^4**
- P(outcome =2) = **C(5,2) 0.6^2 *0.4^3**
- P(outcome =3) = **?**

Probabilities

- **Repeated coin flip problem**
- Assume **the random variable is the count of occurrences of heads** in 5 coin flips. For example:
- TTTTT yields outcome 0
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- What is the probability of an outcome 0?
- P(outcome=0) = **C(5,0) 0.6^0 *0.4^5**
- P(outcome=1) = **C(5,1) 0.6^1 *0.4^4**
- P(outcome =2) = **C(5,2) 0.6^2 *0.4^3**
- P(outcome =3) = **C(5,3) 0.6^3 *0.4^2**
- …

Expected value of the random variable

Definition: The expected value of the random variable X(s) on the sample space is equal to:

$$E(X) = \sum_{s \in S} p(s) X(s)$$

Example: roll of a dice

- Outcomes: ?

Expected value and variance

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Example: roll of a dice

- Outcomes: 1 2 3 4 5 6
- Expected value:

  E(X) = ?

Expected value and variance

Definition: The expected value of the random variable X(s) on the sample space is equal to:

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Example: roll of a dice

- Outcomes: 1 2 3 4 5 6
- Expected value:

E(X) = 1*1/6 + 2*1/6+3*1/6 + 4*1/6 + 5*1/6 +6* 1/6 = 7/2

Expected value

Example:

Flip a fair coin 3 times (trial). A random variable X is the number of heads in the 3 flips. What is the expected value of X?

Answer:

Possible outcomes of a trial (sample space):

= ?

Expected value

Example:

Flip a fair coin 3 times. A random variable X is the number of heads in the 3 flips. What is the expected value of X?

Answer:

Possible outcomes of a trial (sample space):

= {HHH HHT HTH THH HTT THT TTH TTT}

?

Expected value

Example:

Flip a fair coin 3 times. A random variable X is the number of heads in the 3 flips. What is the expected value of X?

Answer:

Possible outcomes of a trial (sample space):

= {HHH HHT HTH THH HTT THT TTH TTT}

HHH	HHT	HTH	THH	HTT	THT	TTH	TTT
3	**2**	**2**	**2**	**1**	**1**	**1**	**0**
1x	**3x**			**3x**			**1x**

E(X) = ?

Expected value

Example:

Flip a fair coin 3 times. A random variable X is the number of heads in the 3 flips. What is the expected value of X?

Answer:

Possible outcomes of a trial (sample space):

= {HHH HHT HTH THH HTT THT TTH TTT}

3 2 2 2 1 1 1 0

1x 3x 3x 1x

E(X) = 1/8 (1*3 + 3*2 +3*1 +1*0) = 12/8= 3/2

½*½*½

Expected value

- **Theorem:** If Xi for i=1,2,3, n with n being a positive integer, are random variables on S, and a and b are real numbers then:
  - E(X1+X2+ ... Xn) = E(X1)+E(X2) + ... E(Xn)
  - E(aX+b) = aE(X) +b

Expected value

Example:

- Roll a pair of dices. What is the expected value of the sum of outcomes?
- **Approach 1:**
- Outcomes: (1,1) (1,2) (1,3) .... (6,1)... (6,6)

  2 3 4 7 12

Expected value: 1/36 (2*1 + ...) = 7

- **Approach 2 (theorem):**
- E(X1+X2) = E(X1) +E(X2)
- E(X1) =7/2 E(X2) = 7/2
- E(X1+X2) = 7