# CS 441 Discrete Mathematics for CS
# Lecture 31

# Probabilities

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## Course administration

- **Homework assignment 10 is out**
  - **due next week on Friday, April 14, 2006**

**Course web page:**

http://www.cs.pitt.edu/~milos/courses/cs441/

# Probabilities

**Axioms of the probability theory:**

- Probability of a discrete outcome (of a random variable) is:
  - $0 <= p(s) <= 1$
- Sum of probabilities over all outcomes is = 1
- For any two events E1 and E2 holds:
  P(E1 U E2) = P(E1) + P(E2) – P(E1 and E2)

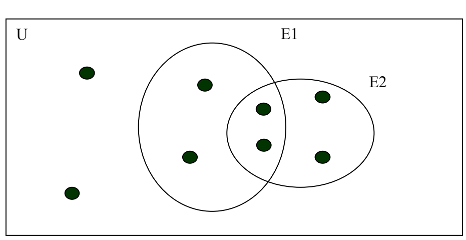


# Probabilities

Let E1 and E2 be two events in the sample space S. Then:

- P(E1 U E2) = P(E1) + P(E2) – P(E1 and E2)

- This is an example of the inclusion-exclusion principle



## Probabilities

**Definition:** A function p: S → [0,1] satisfying axiom conditions is called **a probability distribution.**

A **uniform distribution** is a special case: it assigns an equal probability to each outcome.

**Example:** a biased coin.

- Probability of head 0.6, probability of a tail 0.4

**Probability distribution:**

- Head → 0.6 The sum of the two probabilities sums to 1
- Tail → 0.4



## Conditional probability

**Definition:** Let E and F be two events such that P(F) > 0. The **conditional probability** of E given F

- P(E|F) = P(E and F) / P(F)

- **Example:**
- What is the probability that a family has two boys given that they have at least one boy. Assume the probability of having a girl or a boy is equal.
- **Possibilities. ?**

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- **P(BB) = ¼**
- **P(one boy) = ?**

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- **Possibilities. BB BG GB GG**
- **P(BB) = ¼**
- **P(one boy) = ¾**
- **P(BB|given a boy) = (¼ )/ (¾) = 1/3**

## Independence

**Definition:** The events E and F are said to be **independent** if:

- P(E and F) = P(E)*P(F)

**Example.** Assume that E denotes the family has three children of both sexes and F the fact that the family has at most one boy. Are E and F independent?

- All combos =



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- All combos = {BBB, BBG, BGB, GBB,BGG,GBG,GGB,GGG} the number of elements = ?

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- All combos = {BBB, BBG, BGB, GBB,BGG,GBG,GGB,GGG} the number of elements = 8
- Both sexes = ?



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- All combos = {BBB, BBG, BGB, GBB,BGG,GBG,GGB,GGG} the number of elements = 8
- Both sexes = {BBG BGB GBB BGG GBG GGB} # = 6
- At most one boy = ?

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- Both sexes = {BBG BGB GBB BGG GBG GGB} # = 6
- At most one boy = {GGG GGB GBG BGG} # = 4
- E and F = ?



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- E and F = {GGB GBG BGG} # = 3
- P(E and F} = ?

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- P(E and F} = 3/8 and P(E)*P(F)= ?



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- Both sexes = {BBG BGB GBB BGG GBG GGB} # = 6
- At most one boy = {GGG GGB GBG BGG} # = 4
- E and F = {GGB GBG BGG} # = 3
- P(E and F} = 3/8 and P(E)*P(F)= 4/8 6/8 = 3/8
- **The two probabilities are equal → E and F are independent**

## Probability distribution

- **Definition: A random variable** is a function from the sample space of an experiment to the set of real numbers f: S → R. A random variable assigns a number to each possible outcome.

- **The distribution of a random variable X on the sample space S** is a set of pairs (r p(X=r) ) for all r in S where r is the number and p(X=r) is the probability that X takes a value r.



## Bernoulli trial

- **Assume a repeated coin flip**
- P(head) =0.6 and the probability of a tail is 0.4. Each coin flip is independent of the previous.
- What is the probability of seeing:
  - HHHHH - 5 heads in a row
- **P(HHHHH) = ?**

## Probabilities

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- What is the probability of seeing:
    - HHHHH - 5 heads in a row
- **P(HHHHH) = $0.6^5$ =**
    - Assume the outcome is HHTTT
- **P(HHTTT)= ?**



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- **P(HHHHH) = $0.6^5$ =**
    - Assume the outcome is HHTTT
- **P(HHTTT)= $0.6*0.6*0.4^{3}= 0.6^2*0.4^3$**
    - Assume the outcome is TTHHT
- **P(TTHHT)=?**

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- What is the probability of seeing three tails and two heads?

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- What is the probability of seeing three tails and two heads?
- The number of two-head-three tail combinations?



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- **P(TTHHT)=$0.4^2*0.6^2*0.4 = 0.6^2*0.4^3$**
- What is the probability of seeing three tails and two heads?
- The number of two-head-three tail combinations = **C(2,5)**
- P(two-heads-three tails) = **C(2,5) *$0.6^2 * 0.4^3$**

## Probabilities

- **Assume a variant of a repeated coin flip problem**
- The space of possible outcomes is the count of occurrences of heads in 5 coin flips. For example:
- TTTTT yields outcome 0
- HTTTT or TTHTT yields 1
- HTHHT yields 3 …

- What is the probability of an outcome 0?
- P(outcome=0) = $\mathbf{0.6^0 * 0.4^5}$
- P(outcome=1) =?



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- P(outcome=0) = $\mathbf{0.6^0 * 0.4^5}$
- P(outcome=1) = $\mathbf{C(1,5)\ 0.6^1 * 0.4^4}$
- P(outcome =2) = **?**



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- P(outcome =3) = **?**

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- P(outcome=0) = $\mathbf{0.6^0 * 0.4^5}$
- P(outcome=1) = **C(1,5)** $\mathbf{0.6^1 * 0.4^4}$
- P(outcome =2) = **C(2,5)** $\mathbf{0.6^2 * 0.4^3}$
- P(outcome =3) = **C(3,5)** $\mathbf{0.6^3 * 0.4^2}$
- …