Probabilities

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Course administration

- Homework assignment 10 is out
  - due next week on Friday, April 14, 2006

Course web page:
http://www.cs.pitt.edu/~milos/courses/cs441/
Probabilities

Axioms of the probability theory:

- Probability of a discrete outcome (of a random variable) is:
  - \( 0 \leq p(s) \leq 1 \)

- Sum of probabilities over all outcomes is \( = 1 \)

- For any two events \( E_1 \) and \( E_2 \) holds:
  \[
  P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \text{ and } E_2)
  \]

Let \( E_1 \) and \( E_2 \) be two events in the sample space \( S \). Then:

- \( P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \text{ and } E_2) \)

- This is an example of the inclusion-exclusion principle
Probabilities

**Definition:** A function \( p: S \rightarrow [0,1] \) satisfying axiom conditions is called a **probability distribution**.

A **uniform distribution** is a special case: it assigns an equal probability to each outcome.

**Example:** a biased coin.
- Probability of head 0.6, probability of a tail 0.4

**Probability distribution:**
- Head \( \rightarrow 0.6 \) The sum of the two probabilities sums to 1
- Tail \( \rightarrow 0.4 \)

Conditional probability

**Definition:** Let \( E \) and \( F \) be two events such that \( P(F) > 0 \). The **conditional probability** of \( E \) given \( F \)

\[
P(E|F) = \frac{P(E \text{ and } F)}{P(F)}\]

**Example:**
- What is the probability that a family has two boys given that they have at least one boy. Assume the probability of having a girl or a boy is equal.

**Possibilities.** ?
Conditional probability

**Definition:** Let E and F be two events such that P(F) > 0. The *conditional probability* of E given F

- \( P(E|F) = \frac{P(E \text{ and } F)}{P(F)} \)

**Example:**
- What is the probability that a family has two boys given that they have at least one boy. Assume the probability of having a girl or a boy is equal.
- **Possibilities.** BB BG GB GG
- \( P(BB) = \frac{1}{4} \)
- \( P(\text{one boy}) = ? \)
Conditional probability

**Definition:** Let $E$ and $F$ be two events such that $P(F) > 0$. The **conditional probability** of $E$ given $F$

- $P(E | F) = \frac{P(E \text{ and } F)}{P(F)}$

**Example:**
- What is the probability that a family has two boys given that they have at least one boy. Assume the probability of having a girl or a boy is equal.
- **Possibilities.** BB BG GB GG
- $P(BB) = \frac{1}{4}$
- $P(\text{one boy}) = \frac{3}{4}$
- $P(BB | \text{given a boy}) = \frac{(\frac{1}{4})}{(\frac{3}{4})} = \frac{1}{3}$
Independence

Definition: The events $E$ and $F$ are said to be independent if:

- $P(E \text{ and } F) = P(E) \times P(F)$

Example. Assume that $E$ denotes the family has three children of both sexes and $F$ the fact that the family has at most one boy. Are $E$ and $F$ independent?

- All combos = 

\{BBB, BBG, BGB, GBB, BGG, GBG, GGB, GGG\}

the number of elements = ?
Independence

**Definition:** The events $E$ and $F$ are said to be **independent** if:

- $P(E \text{ and } F) = P(E) \times P(F)$

**Example.** Assume that $E$ denotes the family has three children of both sexes and $F$ the fact that the family has at most one boy. Are $E$ and $F$ independent?

- All combos = \{BBB, BBG, BGB, GBB, BGG, GGB, GGB, GGG\}
  - the number of elements = 8
- Both sexes = ?


**Example.** Assume that $E$ denotes the family has three children of both sexes and $F$ the fact that the family has at most one boy. Are $E$ and $F$ independent?

- All combos = \{BBB, BBG, BGB, GBB, BGG, GGB, GGB, GGG\}
  - the number of elements = 8
- Both sexes = \{BBG, BGB, GBB, BGG, GGB, GGB\}
  - $\# = 6$
- At most one boy = ?
Independence

**Definition:** The events E and F are said to be **independent** if:

- \( P(E \text{ and } F) = P(E) \cdot P(F) \)

**Example.** Assume that E denotes the family has three children of both sexes and F the fact that the family has at most one boy. Are E and F independent?

- All combos = \{BBB, BBG, BGB, GBB, BGG, GBG, GGB, GGG\}  
  the number of elements = 8
- Both sexes = \{BBG, BGB, GBB, BGG, GBG, GGB\}  
  # = 6
- At most one boy = \{GGG, GGB, GBG, BGG\}  
  # = 4
- E and F = ?  
  # = ?

\[
P(E \text{ and } F) = P(E) \cdot P(F)
\]
Independence

**Definition:** The events $E$ and $F$ are said to be independent if:

- $P(E \text{ and } F) = P(E)P(F)$

**Example.** Assume that $E$ denotes the family has three children of both sexes and $F$ the fact that the family has at most one boy. Are $E$ and $F$ independent?

- All combos = \{BBB, BBG, BGB, GBB, BGG, GBG, GGB, GGG\}
  the number of elements = 8
- Both sexes = \{BBG, BGB, GBB, BGG, GBG, GGB\} # = 6
- At most one boy = \{GGG, GGB, GBG, BGG\} # = 4
- $E$ and $F$ = \{GGB, GBG, BGG\} # = 3
- $P(E \text{ and } F) = \frac{3}{8}$ and $P(E)P(F) = \frac{4}{8} \cdot \frac{6}{8} = \frac{3}{8}$

The two probabilities are equal $\rightarrow$ $E$ and $F$ are independent
Probability distribution

• **Definition: A random variable** is a function from the sample space of an experiment to the set of real numbers \( S \rightarrow \mathbb{R} \). A random variable assigns a number to each possible outcome.

• **The distribution of a random variable** \( X \) **on the sample space**
  \( S \) is a set of pairs \((r, p(X=r))\) for all \( r \) in \( S \) where \( r \) is the number and \( p(X=r) \) is the probability that \( X \) takes a value \( r \).

Bernoulli trial

• **Assume a repeated coin flip**
• \( P(\text{head}) = 0.6 \) and the probability of a tail is 0.4. Each coin flip is independent of the previous.
• What is the probability of seeing:
  • HHHHH - 5 heads in a row
• \( P(\text{HHHHH}) = ? \)
Probabilities

• Assume a repeated coin flip
• P(head) = 0.6 and the probability of a tail is 0.4. Each coin flip is independent of the previous.
• What is the probability of seeing:
  • HHHHH - 5 heads in a row
• \( P(\text{HHHHH}) = 0.6^5 = \)
  • Assume the outcome is HHTTT
• \( P(\text{HHTTT}) =? \)
Probabilities

• Assume a repeated coin flip
• \( P(\text{head}) = 0.6 \) and the probability of a tail is 0.4. Each coin flip is independent of the previous.
• What is the probability of seeing:
  • HHHHH - 5 heads in a row
  • \( P(\text{HHHHH}) = 0.6^5 = \)
  • Assume the outcome is HHTTT
  • \( P(\text{HHTTT}) = 0.6^3 \cdot 0.4^2 = 0.6^2 \cdot 0.4^3 \)
  • Assume the outcome is TTHHT
  • \( P(\text{TTHHT}) = 0.4^2 \cdot 0.6^2 \cdot 0.4 = 0.6^2 \cdot 0.4^3 \)

• What is the probability of seeing three tails and two heads?
Probabilities

- Assume a repeated coin flip
- \( P(\text{head}) = 0.6 \) and the probability of a tail is 0.4. Each coin flip is independent of the previous.
- What is the probability of seeing:
  - HHHHH - 5 heads in a row
- \( P(\text{HHHHH}) = 0.6^5 = \)
  - Assume the outcome is HHTTT
- \( P(\text{HHTTT}) = 0.6 \times 0.6 \times 0.4^3 = 0.6^2 \times 0.4^3 \)
  - Assume the outcome is TTHHT
- \( P(\text{TTHHT}) = 0.4^2 \times 0.6^2 \times 0.4 = 0.6^2 \times 0.4^3 \)
- What is the probability of seeing three tails and two heads?
- The number of two-head-three tail combinations = \( \binom{2}{5} \)
- \( P(\text{two-heads-three tails}) = \binom{2}{5} \times 0.6^2 \times 0.4^3 \)
Probabilities

• **Assume a variant of a repeated coin flip problem**
• The space of possible outcomes is the count of occurrences of heads in 5 coin flips. For example:
  • TTTTT yields outcome 0
  • HTTTT or TTHTT yields 1
  • HTHHT yields 3 …

• What is the probability of an outcome 0?
  • $P(\text{outcome}=0) = 0.6^0 \cdot 0.4^5$
  • $P(\text{outcome}=1) =$?
Probabilities

• **Assume a variant of a repeated coin flip problem**
• The space of possible outcomes is the count of occurrences of heads in 5 coin flips. For example:
  • TTTTT yields outcome 0
  • HTTTT or TTHTT yields 1
  • HTHHT yields 3 …

• What is the probability of an outcome 0?
  • \( P(\text{outcome}=0) = 0.6^0 \cdot 0.4^5 \)
  • \( P(\text{outcome}=1) = \binom{1,5} 0.6^1 \cdot 0.4^4 \)
  • \( P(\text{outcome}=2) = ? \)
  • \( P(\text{outcome}=3) = ? \)
Probabilities

• Assume a variant of a repeated coin flip problem
• The space of possible outcomes is the count of occurrences of heads in 5 coin flips. For example:
  • TTTTT yields outcome 0
  • HTTTT or TTHTT yields 1
  • HTHHT yields 3 …

• What is the probability of an outcome 0?
  • P(outcome=0) = 0.6^0 * 0.4^5
  • P(outcome=1) = C(1, 5) * 0.6^1 * 0.4^4
  • P(outcome =2) = C(2, 5) * 0.6^2 * 0.4^3
  • P(outcome =3) = C(3, 5) * 0.6^3 * 0.4^2
  • …