CS 441 Discrete Mathematics for CS
Lecture 30

Probabilities

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Course administration

- **Homework assignment**
  - **no assignment is due this week**
- **Recitation today**
  - **Midterm solutions**

Course web page:

http://www.cs.pitt.edu/~milos/courses/cs441/

Probability

Discrete probability theory.

- Used to compute the odds of seeing some outcomes. Related to counting when the outcomes are equally likely.

Example: Coin flip

- Assume 2 outcomes (head and tail) and each of them is equally likely
- Odds: 50%, 50%
- the probability of seeing:
    - a head is 0.5
    - a tail is 0.5

Probabilities

- **Experiment:** a procedure that yields one of the possible outcomes
- **Sample space:** a set of possible outcomes
- **Event:** a subset of possible outcomes (E is a subset of S)
- **Assuming the outcomes are equally likely, the probability of an event E, defined by a subset of outcomes from the sample space S is**
    - **P(Event)= |E| / |S|**
- The cardinality of the subset divided by the cardinality of the sample space.

Probabilities

Example 1:

- A box with 4 red balls and 6 blue balls. What is the probability that we pull the red ball out.
- **P(E) =?**

Probabilities

Example 1:

- A box with 4 red balls and 6 blue balls. What is the probability that we pull the red ball out.
- **P(E) =4/10 =0.4**

Probabilities

Example 2:

- roll of two dices.
- What is the probability that the outcome is 7.
- Possible outcomes: ?

Probabilities

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- roll of two dices.
- What is the probability that the outcome is 7.
- Possible outcomes:
- (1,6) (2,6) …(6,1),…(6,6) total: 36

- Outcomes leading to 7?

Probabilities

Example 2:

- roll of two dices.
- What is the probability that the outcome is 7.
- Possible outcomes:
- (1,6) (2,6) …(6,1),…(6,6)   total: 36

- Outcomes leading to
- (1,6) (2,5) …(6,1)   total: 6
- P(sum=7)= ?

Probabilities

Example 2:

- roll of two dices.
- What is the probability that the outcome is 7.
- Possible outcomes:
- (1,6) (2,6) …(6,1),…(6,6)   total: 36

- Outcomes leading to
- (1,6) (2,5) …(6,1)   total: 6
- P(sum=7)= 6/36 = 1/6

Probabilities

More complex:

- Odd of winning a lottery: 6 numbers out of 40.
- Total number of outcomes: ?

Probabilities

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- Odd of winning a lottery: 6 numbers out of 40.
- Total number of outcomes:
  - C(40,6) = …
- Probability of winning: ?
  - P(E) =?

Probabilities

More complex:

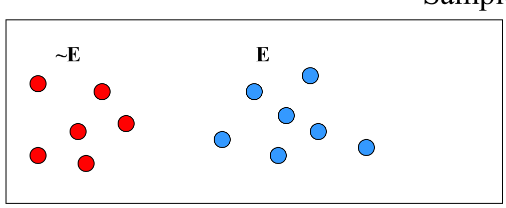
- Odd of winning a lottery: 6 numbers out of 40.
- Total number of outcomes:
  - C(40,6) = …
- Probability of winning: ?
  - P(E) =1/C(40,6)= 34! 6! / 40! = 3,838,380

Probabilities

Theorem: Let E be an event and ~E its complement with regard to S. Then:

- P(~E) = 1 – P(E)

Sample space



Proof.

Probabilities

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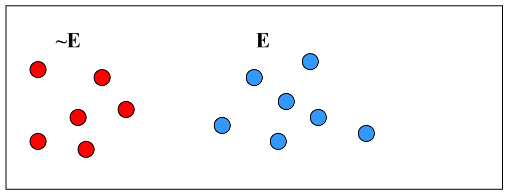
- P(~E) = 1 – P(E)

Sample space



Proof.

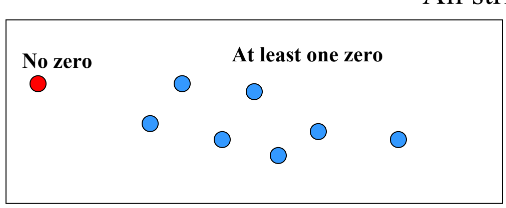
P(~E) = (|S|- |E|) /| S | = 1- |E| / |S|

Probabilities

Example:

- 10 randomly generated bits. What is the probability that there is at least one zero in the string.

All strings



- Event: seeing all zero string
- ~Event: seeing at least one string

Probabilities

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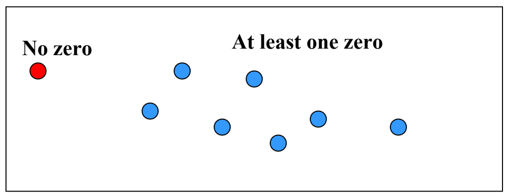
- 10 randomly generated bits. What is the probability that there is at least one zero in the string.

All strings



- Event: seeing all zero string   P(E) =?
- ~Event: seeing at least one string

Probabilities

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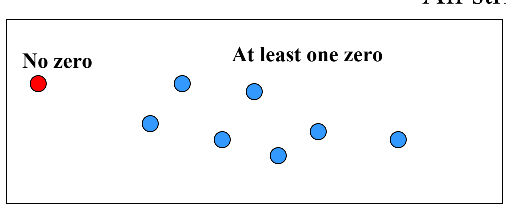
All strings

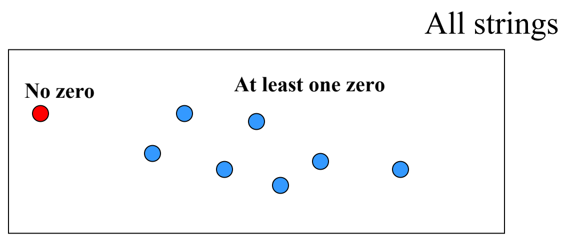


- Event: seeing all zero string $P(E) = 1/2^{10}$
- ~Event: seeing at least one string $P(\sim E) = ?$

Probabilities

Example:

- 10 randomly generated bits. What is the probability that there is at least one zero in the string.



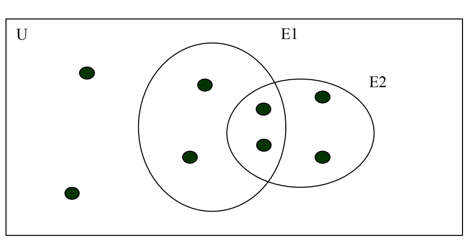
- Event: seeing all zero string $P(E) = 1/2^{10}$
- ~Event: seeing at least one string $P(\sim E) = 1 - P(E) = 1 - 1/2^{10}$

Probabilities

Theorem. Let E1 and E2 be two events in the sample space S. Then:

- P(E1 U E2) = P(E1) + P(E2) – P(E1 and E2)

- This is an example of the inclusion-exclusion principle



Probabilities

Theorem. Let E1 and E2 be two events in the sample space S. Then:

- P(E1 U E2) = P(E1) + P(E2) – P(E1 and E2)

Example: Probability that a positive integer <= 100 is divisible either by 2 or 5.

- E1: the integer is divisible by 2, E2: the integer is divisible by 5
- P(E1)= ?

Probabilities

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- P(E1 U E2) = P(E1) + P(E2) – P(E1 and E2)

Example: Probability that a positive integer <= 100 is divisible either by 2 or 5.

- P(E1)= 50/100
- P(E2)= ?

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Example: Probability that a positive integer <= 100 is divisible either by 2 or 5.

- P(E1)= 50/100
- P(E2)= 20/100
- P(E1 and E2) = ?

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- P(E1 and E2) = 10/100
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Example: Probability that a positive integer <= 100 is divisible either by 2 or 5.

- P(E1)= 50/100
- P(E2)= 20/100
- P(E1 and E2) = 10/100
- P(E1 U E2) = (5+2-1)/10 = 6/10

Probabilities

- Assumption applied so far:
  - **the probabilities of each outcome are equally likely.**
- However in many cases outcome may not be equally likely.

Example: a biased coin or a biased dice.

- Probability of head 0.6, probability of a tail 0.4.

Probabilities

Axioms of the probability theory:

- Probability of a discrete outcome is:
  - $0 <= p(s) <= 1$
- Sum of probabilities over all outcomes is = 1
- For any two events E1 and E2 holds:
  P(E1 U E2) = P(E1) + P(E2) – P(E1 and E2)

Probabilities

Definition: A function p: S → [0,1] satisfying a condition is called **a probability distribution.**

A uniform distribution is a special case: assigns an equal probability to each outcome.

Example: a biased coin.

- Probability of head 0.6, probability of a tail 0.4

Probability distribution:

- Head → 0.6 The sum of the two probabilities sums to 1
- Tail → 0.4

Conditional probability

Definition: Let E and F be two events such that P(F) > 0. The **conditional probability** of E given F

- P(E|F) = P(E and F) / P(F)

- **Example:**
- What is the probability that a family has two boys given that they have at least one boy. Assume the probability of having a girl or a boy is equal.
- **Possibilities. ?**

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- **Possibilities. BB BG GB GG**
- **P(BB) = ?**

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- **P(BB) = ¼**
- **P(one boy) = ?**

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- **P(one boy) = ¾**
- **P(BB|given a boy) = ?**

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Example:

- What is the probability that a family has two boys given that they have at least one boy. Assume the probability of having a girl or a boy is equal.
- **Possibilities. BB BG GB GG**
- **P(BB) = ¼**
- **P(one boy) = ¾**
- **P(BB|given a boy) = ¼ / ¾ = 1/3**

Independence

Definition: The events E and F are said to be **independent** if:

- P(E and F) = P(E)P(F)

Example. Assume that E denotes the family has three children of both sexes and F the fact that the family has at most one boy. Are E and F independent?

- All combos =

Independence

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- All combos = {BBB, BBG, BGB, GBB,BGG,GBG,GGB,GGG} the number of elements = ?

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- All combos = {BBB, BBG, BGB, GBB,BGG,GBG,GGB,GGG} the number of elements = 8
- Both sexes = ?

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- All combos = {BBB, BBG, BGB, GBB,BGG,GBG,GGB,GGG} the number of elements = 8
- Both sexes = {BBG BGB GBB BGG GBG GGB} # = 6
- At most one boy = ?

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- Both sexes = {BBG BGB GBB BGG GBG GGB} # = 6
- At most one boy = {GGG GGB GBG BGG} # = 4
- E and F = ?

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- Both sexes = {BBG BGB GBB BGG GBG GGB} # = 6
- At most one boy = {GGG GGB GBG BGG} # = 4
- E and F = {GGB GBG BGG} # = 3
- P(E and F} = ?

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- Both sexes = {BBG BGB GBB BGG GBG GGB} # = 6
- At most one boy = {GGG GGB GBG BGG} # = 4
- E and F = {GGB GBG BGG} # = 3
- P(E and F} = 3/8 and P(E)*P(F)= ?

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- All combos = {BBB, BBG, BGB, GBB,BGG,GBG,GGB,GGG} the number of elements = 8
- Both sexes = {BBG BGB GBB BGG GBG GGB} # = 6
- At most one boy = {GGG GGB GBG BGG} # = 4
- E and F = {GGB GBG BGG} # = 3
- P(E and F} = 3/8 and P(E)*P(F)= 4/8 6/8 = 3/8
- **The two probabilities are equal → E and F are independent**