Probabilities

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Course administration

• Homework assignment
  – no assignment is due this week

• Recitation today
  – Midterm solutions

Course web page:
http://www.cs.pitt.edu/~milos/courses/cs441/
Probability

Discrete probability theory.
• Used to compute the odds of seeing some outcomes. Related to counting when the outcomes are equally likely.

Example: Coin flip
• Assume 2 outcomes (head and tail) and each of them is equally likely
• Odds: 50%, 50%
• the probability of seeing:
  • a head is 0.5
  • a tail is 0.5

Probabilities
• **Experiment**: a procedure that yields one of the possible outcomes
• **Sample space**: a set of possible outcomes
• **Event**: a subset of possible outcomes (E is a subset of S)
• **Assuming the outcomes are equally likely**, the probability of an event E, defined by a subset of outcomes from the sample space S is
  • \( P(\text{Event}) = \frac{|E|}{|S|} \)

• The cardinality of the subset divided by the cardinality of the sample space.
Probabilities

Example 1:
• A box with 4 red balls and 6 blue balls. What is the probability that we pull the red ball out.

• \( P(E) =? \)

\[ P(E) = \frac{4}{10} = 0.4 \]
Probabilities

Example 2:
• roll of two dices.
• What is the probability that the outcome is 7.
• Possible outcomes: ?

Probabilities

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• roll of two dices.
• What is the probability that the outcome is 7.
• Possible outcomes:
  • (1,6) (2,6) …(6,1),…(6,6) total: 36

• Outcomes leading to 7?
Probabilities

Example 2:
• roll of two dices.
• What is the probability that the outcome is 7.
• Possible outcomes:
  • (1,6) (2,6) …(6,1),…(6,6) total: 36

• Outcomes leading to
  • (1,6) (2,5) …(6,1) total: 6
  • P(sum=7)= \( \frac{6}{36} = \frac{1}{6} \)
Probabilities

More complex:
• Odd of winning a lottery: 6 numbers out of 40.
• Total number of outcomes: ?

Probabilities

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• Odd of winning a lottery: 6 numbers out of 40.
• Total number of outcomes:
  • C(40,6) = …
• Probability of winning: ?
  • P(E) =?
Probabilities

More complex:
• Odd of winning a lottery: 6 numbers out of 40.
• Total number of outcomes:
  • $C(40, 6) = \ldots$
• Probability of winning: ?
  • $P(E) = \frac{1}{C(40, 6)} = \frac{34!}{6!} / 40! = 3,838,380$

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Probabilities

**Theorem:** Let $E$ be an event and $\sim E$ its complement with regard to $S$. Then:
• $P(\sim E) = 1 - P(E)$

**Proof.**
Probabilities

**Theorem:** Let $E$ be an event and $\sim E$ its complement with regard to $S$. Then:
- $P(\sim E) = 1 - P(E)$

**Proof.**

$$P(\sim E) = \frac{|S| - |E|}{|S|} = 1 - \frac{|E|}{|S|}$$

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Probabilities

**Example:**
- 10 randomly generated bits. What is the probability that there is at least one zero in the string.

**Sample space**

- Event: seeing all zero string
- $\sim$Event: seeing at least one string
Probabilities

Example:
• 10 randomly generated bits. What is the probability that there is at least one zero in the string.

• Event: seeing all zero string \( P(E) = \)?
• \( \sim \) Event: seeing at least one string

\[
\begin{array}{c|c|c}
\text{All strings} & \text{No zero} & \text{At least one zero} \\
\end{array}
\]

• Event: seeing all zero string \( P(E) = \frac{1}{2^{10}} \)
• \( \sim \) Event: seeing at least one string \( P(\sim E) = \)?
**Probabilities**

**Example:**
- 10 randomly generated bits. What is the probability that there is at least one zero in the string.

All strings

- Event: seeing all zero string \( P(E) = \frac{1}{2^{10}} \)
- \( \sim \)-Event: seeing at least one string \( P(\sim E) = 1 - P(E) = 1 - \frac{1}{2^{10}} \)

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**Theorem.** Let \( E_1 \) and \( E_2 \) be two events in the sample space \( S \). Then:
- \( P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \text{ and } E_2) \)

- This is an example of the inclusion-exclusion principle
**Probabilities**

**Theorem.** Let $E_1$ and $E_2$ be two events in the sample space $S$. Then:

- $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \text{ and } E_2)$

**Example:** Probability that a positive integer $\leq 100$ is divisible either by 2 or 5.
- $E_1$: the integer is divisible by 2, $E_2$: the integer is divisible by 5
- $P(E_1) =$ ?
- $P(E_2) =$ ?
Probabilities

**Theorem.** Let $E_1$ and $E_2$ be two events in the sample space $S$. Then:

- $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \text{ and } E_2)$

**Example:** Probability that a positive integer $\leq 100$ is divisible either by 2 or 5.
- $P(E_1) = 50/100$
- $P(E_2) = 20/100$
- $P(E_1 \text{ and } E_2) = ?$

- $P(E_1 U E_2) = ?$
Probabilities

Theorem. Let $E_1$ and $E_2$ be two events in the sample space $S$. Then:

- $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \text{ and } E_2)$

Example: Probability that a positive integer $\leq 100$ is divisible either by 2 or 5.

- $P(E_1) = 50/100$
- $P(E_2) = 20/100$
- $P(E_1 \text{ and } E_2) = 10/100$
- $P(E_1 \cup E_2) = (5+2-1)/10 = 6/10$

Probabilities

- Assumption applied so far:
  - the probabilities of each outcome are equally likely.
- However in many cases outcome may not be equally likely.

Example: a biased coin or a biased dice.

- Probability of head 0.6, probability of a tail 0.4.
Probabilities

Axioms of the probability theory:
• Probability of a discrete outcome is:
  • \(0 \leq p(s) \leq 1\)

• Sum of probabilities over all outcomes is = 1

• For any two events \(E_1\) and \(E_2\) holds:
  \(P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \text{ and } E_2)\)

Definition: A function \(p: S \rightarrow [0,1]\) satisfying a condition is called a probability distribution.

A uniform distribution is a special case: assigns an equal probability to each outcome.

Example: a biased coin.
• Probability of head 0.6, probability of a tail 0.4

Probability distribution:
• Head \(\rightarrow\) 0.6 The sum of the two probabilities sums to 1
• Tail \(\rightarrow\) 0.4
Conditional probability

**Definition:** Let $E$ and $F$ be two events such that $P(F) > 0$. The **conditional probability** of $E$ given $F$

- $P(E|F) = \frac{P(E \text{ and } F)}{P(F)}$

- **Example:**
  - What is the probability that a family has two boys given that they have at least one boy. Assume the probability of having a girl or a boy is equal.
  - **Possibilities.** BB BG GB GG
  - $P(BB) = \ ?$
Conditional probability

**Definition:** Let E and F be two events such that \( P(F) > 0 \). The **conditional probability** of E given F

- \( P(E|F) = \frac{P(E \text{ and } F)}{P(F)} \)

**Example:**
- What is the probability that a family has two boys given that they have at least one boy. Assume the probability of having a girl or a boy is equal.
- **Possibilities.** BB BG GB GG
- \( P(BB) = \frac{1}{4} \)
- \( P(\text{one boy}) = ? \)

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**Conditional probability**

**Definition:** Let E and F be two events such that \( P(F) > 0 \). The **conditional probability** of E given F

- \( P(E|F) = \frac{P(E \text{ and } F)}{P(F)} \)

**Example:**
- What is the probability that a family has two boys given that they have at least one boy. Assume the probability of having a girl or a boy is equal.
- **Possibilities.** BB BG GB GG
- \( P(BB) = \frac{1}{4} \)
- \( P(\text{one boy}) = \frac{3}{4} \)
- \( P(BB|\text{given a boy}) = ? \)
**Conditional probability**

**Definition:** Let E and F be two events such that P(F) > 0. The **conditional probability** of E given F
- \( P(E|F) = \frac{P(E \text{ and } F)}{P(F)} \)

**Example:**
- What is the probability that a family has two boys given that they have at least one boy. Assume the probability of having a girl or a boy is equal.
- **Possibilities.** BB BG GB GG
- \( P(BB) = \frac{1}{4} \)
- \( P(\text{one boy}) = \frac{3}{4} \)
- \( P(BB|\text{given a boy}) = \frac{1}{4} / \frac{3}{4} = \frac{1}{3} \)

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**Independence**

**Definition:** The events E and F are said to be **independent** if:
- \( P(E \text{ and } F) = P(E)P(F) \)

**Example.** Assume that E denotes the family has three children of both sexes and F the fact that the family has at most one boy. Are E and F independent?
- All combos =
Independence

**Definition:** The events $E$ and $F$ are said to be **independent** if:

- $P(E \text{ and } F) = P(E)P(F)$

**Example.** Assume that $E$ denotes the family has three children of both sexes and $F$ the fact that the family has at most one boy. Are $E$ and $F$ independent?

- All combos $= \{\text{BBB, BBG, BGB, GBB, BGG, GGB, GBB, GGG}\}$
  - the number of elements $= ?$

- Both sexes $= ?$
Independence

Definition: The events $E$ and $F$ are said to be independent if:

• $P(E \text{ and } F) = P(E)P(F)$

Example. Assume that $E$ denotes the family has three children of both sexes and $F$ the fact that the family has at most one boy.

Are $E$ and $F$ independent?

• All combos = \{BBB, BBG, BGB, GBB, BGG, GBG, GGB, GGG\} the number of elements = 8
• Both sexes = \{BBG, BGB, GBG\} # = 6
• At most one boy = ?

E and $F =$ ?
Independence

**Definition:** The events E and F are said to be independent if:

- \( P(E \text{ and } F) = P(E)P(F) \)

**Example.** Assume that E denotes the family has three children of both sexes and F the fact that the family has at most one boy. Are E and F independent?

- All combos = \{BBB, BBG, BGB, GBB, BGG, GGB, GGG\} the number of elements = 8
- Both sexes = \{BBG, BGB, BGG, GGB\} \( \# = 6 \)
- At most one boy = \{GGG, GGB, GBG, BGG\} \( \# = 4 \)
- E and F = \{GGB, GBG, BG, G\} \( \# = 3 \)
- \( P(E \text{ and } F) = 3/8 \) and \( P(E)*P(F) = ? \)
Independence

**Definition:** The events E and F are said to be **independent** if:

- \( P(E \text{ and } F) = P(E)P(F) \)

**Example.** Assume that E denotes the family has three children of both sexes and F the fact that the family has at most one boy. Are E and F independent?

- All combos = \{BBB, BBG, BGB, GBB, BGG, GBG, GGB, GGG\}
  - the number of elements = 8
- Both sexes = \{BBG, BGB, GBB, BGG, GBG, GGB\}  # = 6
- At most one boy = \{GGG, GGB, GBG, BGG\}  # = 4
- E and F = \{GGB, GBG, BGG\}  # = 3
- \( P(E \text{ and } F) = \frac{3}{8} \)  and  \( P(E)P(F) = \frac{4}{8} \cdot \frac{6}{8} = \frac{3}{8} \)
- **The two probabilities are equal \( \Rightarrow \) E and F are independent**