## CS 441 Discrete Mathematics for CS
## Lecture 3

# Propositional logic
# Equivalences

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## Propositional logic: review

- **Propositional logic:** a formal language for making logical inferences
- **A proposition is** a statement that is either true or false.
- **A compound proposition** can be created from other propositions using logical connectives
- **The truth of a compound proposition** is defined by truth values of elementary propositions and the meaning of connectives.
- **The truth table for a compound proposition:** table with entries (rows) for all possible combinations of truth values of elementary propositions.

# Translation

If you are older than 13 or you are with your parents then you can attend a PG-13 movie.

**Parse:**

- **If ( you are older than 13 or you are with your parents ) then ( you can attend a PG-13 movie)**
  - A= you are older than 13
  - B= you are with your parents
  - C=you can attend a PG-13 movie
- **Translation:** $A \vee B \rightarrow C$

- **Why do we want to do this?**
- **Inference: Assume I know that $A \vee B \rightarrow C$ is a correct statement and both A and B are true. Then we can conclude that C is true as well.**



# Computer representation of True and False

- **We need to encode two values True and False:**
  - **use a bit**
  - a bit represents two possible values:
  - 0 (False) or 1(True)

- A variable that takes on values 0 or 1 is called a **Boolean variable**.

- **Definition**: A **bit string** is a sequence of zero or more bits. The **length** of this string is the number of bits in the string.

# Bitwise operations

- T and F replaced with 1 and 0

| p | q | $p \vee q$ | $p \wedge q$ |
|---|---|---|---|
| 1 | 1 | 1 | 1 |
| 1 | 0 | 1 | 0 |
| 0 | 1 | 1 | 0 |
| 0 | 0 | 0 | 0 |

| p | $\neg p$ |
|---|---|
| 1 | 0 |
| 0 | 1 |



# Bitwise operations

- **Examples:**

$$\begin{array}{r} 1011\ 0011 \\ \vee\ \underline{0110\ 1010} \end{array} \qquad \begin{array}{r} 1011\ 0011 \\ \wedge\ \underline{0110\ 1010} \end{array} \qquad \begin{array}{r} 1011\ 0011 \\ \oplus\ \underline{0110\ 1010} \end{array}$$

# Bitwise operations

- **Examples:**

$$\begin{array}{r} 1011\ 0011 \\ \vee\ \underline{0110\ 1010} \\ 1111\ 1011 \end{array} \qquad \begin{array}{r} 1011\ 0011 \\ \wedge\ \underline{0110\ 1010} \\ \phantom{0} \end{array} \qquad \begin{array}{r} 1011\ 0011 \\ \oplus\ \underline{0110\ 1010} \\ \phantom{0} \end{array}$$



# Bitwise operations

- **Examples:**

$$\begin{array}{r} 1011\ 0011 \\ \vee\ \underline{0110\ 1010} \\ 1111\ 1011 \end{array} \qquad \begin{array}{r} 1011\ 0011 \\ \wedge\ \underline{0110\ 1010} \\ 0010\ 0010 \end{array} \qquad \begin{array}{r} 1011\ 0011 \\ \oplus\ \underline{0110\ 1010} \\ \phantom{0} \end{array}$$

# Bitwise operations

- **Examples:**

| | | |
|---|---|---|
| 1011 0011 | 1011 0011 | 1011 0011 |
| $\vee$ 0110 1010 | $\wedge$ 0110 1010 | $\oplus$ 0110 1010 |
| 1111 1011 | 0010 0010 | 1101 1001 |



# Tautology and Contradiction

- Some propositions are interesting since their values in the truth table are always the same

Definitions:

- A compound proposition that is always true for all possible truth values of the propositions is called a **tautology**.
- A compound proposition that is always false is called a **contradiction**.
- A proposition that is neither a tautology nor contradiction is called a **contingency**.

**Example:** $p \vee \neg p$

| $p$ | $\neg p$ | $p \vee \neg p$ |
|---|---|---|
| T | F | |
| F | T | |

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**Example:** $p \vee \neg p$ is a **tautology.**

| $p$ | $\neg p$ | $p \vee \neg p$ |
|---|---|---|
| T | F | T |
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Example: $p \wedge \neg p$ is a **contradiction.**

| p | ¬p | p ∧ ¬p |
|---|---|---|
| T | F | F |
| F | T | F |



# Equivalence

- We have seen that some of the propositions are equivalent. Their truth values in the truth table are the same.
- Example: $\mathbf{p \to q}$ is equivalent to $\mathbf{\neg q \to \neg p}$ **(contrapositive)**

| p | q | p → q | ¬q → ¬p |
|---|---|---|---|
| T | T | | |
| T | F | | |
| F | T | | |
| F | F | | |

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| p | q | $p \rightarrow q$ | $\neg q \rightarrow \neg p$ |
|---|---|---|---|
| T | T | T | |
| T | F | F | |
| F | T | T | |
| F | F | T | |

- .



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|---|---|---|---|
| T | T | T | T |
| T | F | F | F |
| F | T | T | T |
| F | F | T | T |

- **Equivalent statements** are important for logical reasoning since they can be substituted and can help us to make a logical argument.

# Logical equivalence

- **Definition**: The propositions p and q are called **logically equivalent** if p $\leftrightarrow$ q is a tautology (alternately, if they have the same truth table). The notation **p <=> q** denotes p and q are logically equivalent.

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- $\mathbf{p \rightarrow q}$ is equivalent to $\mathbf{\neg q \rightarrow \neg p}$ **(contrapositive)**

| p | q | $p \rightarrow q$ | $\neg q \rightarrow \neg p$ | $(p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p)$ |
|---|---|---|---|---|
| T | T | T | T | T |
| T | F | F | F | T |
| F | T | T | T | T |
| F | F | T | T | T |

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**Important equivalences:**

- **DeMorgan's Laws:**
- 1) $\neg( p \vee q ) \iff \neg p \wedge \neg q$
- 2) $\neg( p \wedge q ) \iff \neg p \vee \neg q$

**Example:** Negate "The summer in Mexico is cold and sunny" with DeMorgan's Laws

**Solution:** ?



# Equivalence

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**Example of important equivalences**

- **DeMorgan's Laws:**
- 1) $\neg( p \vee q ) \iff \neg p \wedge \neg q$
- 2) $\neg( p \wedge q ) \iff \neg p \vee \neg q$

**Example:** Negate "The summer in Mexico is cold and sunny" with DeMorgan's Laws

**Solution:** "The summer in Mexico is not cold or not sunny."

# Equivalence

**Example of important equivalences**

- **DeMorgan's Laws:**
- 1) $\neg( p \vee q ) \iff \neg p \wedge \neg q$
- 2) $\neg( p \wedge q ) \iff \neg p \vee \neg q$

**To convince us that two propositions are logically equivalent use the truth table**

| p | q | ¬p | ¬q | ¬(p ∨ q) | ¬p ∧ ¬q |
|---|---|---|---|---|---|
| T | T | F | F | | |
| T | F | F | T | | |
| F | T | T | F | | |
| F | F | T | T | | |



# Equivalence

**Example of important equivalences**

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|---|---|---|---|---|---|
| T | T | F | F | F | |
| T | F | F | T | F | |
| F | T | T | F | F | |
| F | F | T | T | T | |

# Equivalence

**Example of important equivalences**

- **DeMorgan's Laws:**
- 1) $\neg( p \vee q ) <=> \neg p \wedge \neg q$
- 2) $\neg( p \wedge q ) <=> \neg p \vee \neg q$

**To convince us that two propositions are logically equivalent use the truth table**

| p | q | ¬p | ¬q | $\neg(p \vee q)$ | $\neg p \wedge \neg q$ |
|---|---|---|---|---|---|
| T | T | F | F | F | F |
| T | F | F | T | F | F |
| F | T | T | F | F | F |
| F | F | T | T | T | T |



# Equivalence

**Example of important equivalences**

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**To convince us that two propositions are logically equivalent use the truth table**

| p | q | ¬p | ¬q | $\neg(p \vee q)$ | $\neg p \wedge \neg q$ |
|---|---|---|---|---|---|
| T | T | F | F | F | F |
| T | F | F | T | F | F |
| F | T | T | F | F | F |
| F | F | T | T | T | T |

# Important logical equivalences

- **Identity**
  - $p \wedge T \iff p$
  - $p \vee F \iff p$
- **Domination**
  - $p \vee T \iff T$
  - $p \wedge F \iff F$
- **Idempotent**
  - $p \vee p \iff p$
  - $p \wedge p \iff p$



# Important logical equivalences

- **Double negation**
  - $\neg(\neg p) \iff p$
- **Commutative**
  - $p \vee q \iff q \vee p$
  - $p \wedge q \iff q \wedge p$
- **Associative**
  - $(p \vee q) \vee r \iff p \vee (q \vee r)$
  - $(p \wedge q) \wedge r \iff p \wedge (q \wedge r)$

# Important logical equivalences

- **Distributive**
  - $p \vee (q \wedge r)$ <=> $(p \vee q) \wedge (p \vee r)$
  - $p \wedge (q \vee r)$ <=> $(p \wedge q) \vee (p \wedge r)$
- **De Morgan**
  - $\neg( p \vee q )$ <=> $\neg p \wedge \neg q$
  - $( p \wedge q )$ <=> $\neg p \vee \neg q$
- **Other useful equivalences**
  - $p \vee \neg p$ <=> T
  - $p \wedge \neg p$ <=> F
  - **$p \rightarrow q$ <=> $(\neg p \vee q)$**



# Using logical equivalences

- **Equivalences can be used in proofs.** A proposition or its part can be transformed using equivalences and some conclusion can be reached.

**Example:** Show that $(p \wedge q) \rightarrow p$ is a tautology.

- Proof: (we must show $(p \wedge q) \rightarrow p$ <=> T)

  $(p \wedge q) \rightarrow p$ <=> $\neg(p \wedge q) \vee p$ Useful

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$(p \wedge q) \rightarrow p \iff \neg(p \wedge q) \vee p$ Useful

$\iff [\neg p \vee \neg q] \vee p$ DeMorgan



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$(p \wedge q) \rightarrow p \iff \neg(p \wedge q) \vee p$ Useful

$\iff [\neg p \vee \neg q] \vee p$ DeMorgan

$\iff [\neg q \vee \neg p] \vee p$ Commutative

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- Proof: (we must show $(p \wedge q) \rightarrow p \;<=>\; T$)

| | | | |
|---|---|---|---|
| $(p \wedge q) \rightarrow p$ | $<=>$ | $\neg(p \wedge q) \vee p$ | Useful |
| | $<=>$ | $[\neg p \vee \neg q] \vee p$ | DeMorgan |
| | $<=>$ | $[\neg q \vee \neg p] \vee p$ | Commutative |
| | $<=>$ | $\neg q \vee [\neg p \vee p]$ | Associative |



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| $(p \wedge q) \rightarrow p$ | $<=>$ | $\neg(p \wedge q) \vee p$ | Useful |
| | $<=>$ | $[\neg p \vee \neg q] \vee p$ | DeMorgan |
| | $<=>$ | $[\neg q \vee \neg p] \vee p$ | Commutative |
| | $<=>$ | $\neg q \vee [\neg p \vee p]$ | Associative |
| | $<=>$ | $\neg q \vee [T]$ | Useful |

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- Proof: (we must show $(p \wedge q) \rightarrow p$ <=> T)

| | | |
|---|---|---|
| $(p \wedge q) \rightarrow p$ | <=> $\neg(p \wedge q) \vee p$ | Useful |
| | <=> $[\neg p \vee \neg q] \vee p$ | DeMorgan |
| | <=> $[\neg q \vee \neg p] \vee p$ | Commutative |
| | <=> $\neg q \vee [\neg p \vee p]$ | Associative |
| | <=> $\neg q \vee [T]$ | Useful |
| | <=> T | Domination |



# Using logical equivalences

- Equivalences can be used in proofs. A proposition or its part can be transformed using equivalences and some conclusion can be reached.

**Example:** Show $(p \wedge q) \rightarrow p$ is a tautology.

- Alternative proof:

| p | q | $p \wedge q$ | $(p \wedge q) \rightarrow p$ |
|---|---|---|---|
| T | T | T | **T** |
| T | F | F | **T** |
| F | T | F | **T** |
| F | F | F | **T** |

# Using logical equivalences

- **Equivalences can be used in proofs.** A proposition or its part can be transformed using equivalences and some conclusion can be reached.

- **Example 2:** Show $(p \rightarrow q) \iff (\neg q \rightarrow \neg p)$

  **Proof:**
- $(p \rightarrow q) \iff (\neg q \rightarrow \neg p)$
- $\iff$ ?



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- Equivalences can be used in proofs. A proposition or its part can be transformed using equivalences and some conclusion can be reached.

- **Example 2:** Show $(p \rightarrow q) \iff (\neg q \rightarrow \neg p)$

  **Proof:**
- $(p \rightarrow q) \iff (\neg q \rightarrow \neg p)$
- $\iff \neg(\neg q) \vee (\neg p)$ Useful
- $\iff$ ?

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  **Proof:**
- $(p \rightarrow q) \iff (\neg q \rightarrow \neg p)$
- $\iff \neg(\neg q) \vee (\neg p)$ Useful
- $\iff q \vee (\neg p)$ Double negation
- $\iff$ ?



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  **Proof:**
- $(p \rightarrow q) \iff (\neg q \rightarrow \neg p)$
- $\iff \neg(\neg q) \vee (\neg p)$ Useful
- $\iff q \vee (\neg p)$ Double negation
- $\iff \neg p \vee q$ Commutative
- $\iff$ ?

# Using logical equivalences

- Equivalences can be used in proofs. A proposition or its part can be transformed using equivalences and some conclusion can be reached.

- **Example 2:** Show $(p \rightarrow q) \Leftrightarrow (\neg q \rightarrow \neg p)$

  **Proof:**

- $(p \rightarrow q) \Leftrightarrow (\neg q \rightarrow \neg p)$
- $\Leftrightarrow \neg(\neg q) \vee (\neg p)$ Useful
- $\Leftrightarrow q \vee (\neg p)$ Double negation
- $\Leftrightarrow \neg p \vee q$ Commutative
- $\Leftrightarrow p \rightarrow q$ Useful

  **End of proof**