Propositional logic: review

- **Propositional logic**: a formal language for making logical inferences
- A **proposition** is a statement that is either true or false.
- A **compound proposition** can be created from other propositions using logical connectives
- The **truth of a compound proposition** is defined by truth values of elementary propositions and the meaning of connectives.
- The **truth table for a compound proposition**: table with entries (rows) for all possible combinations of truth values of elementary propositions.
Translation

If you are older than 13 or you are with your parents then you can attend a PG-13 movie.

Parse:

- **If** (you are older than 13 or you are with your parents) **then** (you can attend a PG-13 movie)
  - A= you are older than 13
  - B= you are with your parents
  - C= you can attend a PG-13 movie
- **Translation**: A ∨ B → C

**Why do we want to do this?**

**Inference**: Assume I know that A ∨ B → C is a correct statement and both A and B are true. Then we can conclude that C is true as well.

Computer representation of True and False

- **We need to encode two values** True and False:
  - use a **bit**
  - a bit represents two possible values:
    - 0 (False) or 1 (True)

- A variable that takes on values 0 or 1 is called a **Boolean** variable.

- **Definition**: A **bit string** is a sequence of zero or more bits. The **length** of this string is the number of bits in the string.
Bitwise operations

• T and F replaced with 1 and 0

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>p ∨ q</th>
<th>p ∧ q</th>
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<tbody>
<tr>
<td>1</td>
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<td>1</td>
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<td>1</td>
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<tbody>
<tr>
<td>1</td>
<td>0</td>
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<tr>
<td>0</td>
<td>1</td>
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</table>

Bitwise operations

• Examples:

\[
\begin{align*}
1011 &\ 0011 &\ 1011 &\ 0011 &\ 1011 &\ 0011 \\
\lor &\ 0110 &\ 1010 &\ 0110 &\ 1010 &\ 0110 &\ 1010
\end{align*}
\]
### Bitwise operations

- **Examples:**

<table>
<thead>
<tr>
<th>Operation</th>
<th>Bit 1</th>
<th>Bit 2</th>
<th>Bit 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lor$</td>
<td>1011 0011</td>
<td>1011 0011</td>
<td>1011 0011</td>
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<tr>
<td>$\land$</td>
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<td>0110 1010</td>
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<tr>
<td>$\oplus$</td>
<td>0110 1010</td>
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<td>0110 1010</td>
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<tr>
<td></td>
<td>1111 1011</td>
<td>0010 0010</td>
<td>1011 0011</td>
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</tbody>
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### Bitwise operations

- **Examples:**

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Bitwise operations

- Examples:

\[
\begin{array}{ccc}
1011 & 0011 & 1011 \\
\lor & 0110 & 1010 \\
1111 & 1011 & 1101 \\
\land & 0110 & 1010 \\
& 0010 & 0010 \\
\oplus & 0110 & 1010 \\
& 1101 & 1001
\end{array}
\]

Tautology and Contradiction

- Some propositions are interesting since their values in the truth table are always the same

Definitions:
- A compound proposition that is always true for all possible truth values of the propositions is called a **tautology**.
- A compound proposition that is always false is called a **contradiction**.
- A proposition that is neither a tautology nor contradiction is called a **contingency**.

Example: \( p \lor \lnot p \)

<table>
<thead>
<tr>
<th>( p )</th>
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<th>( p \lor \lnot p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>F</td>
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Example: \( p \lor \neg p \) is a tautology.

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### Tautology and Contradiction

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Example: \( p \land \neg p \) is a **contradiction**.

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### Equivalence

- We have seen that some of the propositions are equivalent. Their truth values in the truth table are the same.
- Example: \( p \rightarrow q \) is equivalent to \( \neg q \rightarrow \neg p \) (contrapositive)

<table>
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<tr>
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• Equivalent statements are important for logical reasoning since they can be substituted and can help us to make a logical argument.
Logical equivalence

• **Definition:** The propositions $p$ and $q$ are called **logically equivalent** if $p \leftrightarrow q$ is a tautology (alternately, if they have the same truth table). The notation $p \leftrightarrow q$ denotes $p$ and $q$ are logically equivalent.

• Example: $p \rightarrow q$ is equivalent to $\neg q \rightarrow \neg p$ (converse).
Logical equivalence

• **Definition:** The propositions p and q are called logically equivalent if \( p \iff q \) is a tautology (alternately, if they have the same truth table). The notation \( p \iff q \) denotes p and q are logically equivalent.

**Important equivalences:**

• **DeMorgan's Laws:**
  
  1) \( \neg( p \lor q ) \iff \neg p \land \neg q \)
  
  2) \( \neg( p \land q ) \iff \neg p \lor \neg q \)

**Example:** Negate "The summer in Mexico is cold and sunny" with DeMorgan's Laws

**Solution:** "The summer in Mexico is not cold or not sunny."

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To convince us that two propositions are logically equivalent use the truth table

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Important logical equivalences

- **Identity**
  - \( p \land T \leftrightarrow p \)
  - \( p \lor F \leftrightarrow p \)

- **Domination**
  - \( p \lor T \leftrightarrow T \)
  - \( p \land F \leftrightarrow F \)

- **Idempotent**
  - \( p \lor p \leftrightarrow p \)
  - \( p \land p \leftrightarrow p \)

- **Double negation**
  - \( \neg(\neg p) \leftrightarrow p \)

- **Commutative**
  - \( p \lor q \leftrightarrow q \lor p \)
  - \( p \land q \leftrightarrow q \land p \)

- **Associative**
  - \( (p \lor q) \lor r \leftrightarrow p \lor (q \lor r) \)
  - \( (p \land q) \land r \leftrightarrow p \land (q \land r) \)
Important logical equivalences

- **Distributive**
  - \( p \lor (q \land r) \iff (p \lor q) \land (p \lor r) \)
  - \( p \land (q \lor r) \iff (p \land q) \lor (p \land r) \)

- **De Morgan**
  - \( \neg (p \lor q) \iff \neg p \land \neg q \)
  - \( (p \land q) \iff \neg p \lor \neg q \)

- **Other useful equivalences**
  - \( p \lor \neg p \iff T \)
  - \( p \land \neg p \iff F \)
  - \( p \rightarrow q \iff (\neg p \land q) \)

Using logical equivalences

- **Equivalences can be used in proofs.** A proposition or its part can be transformed using equivalences and some conclusion can be reached.

**Example:** Show that \((p \land q) \rightarrow p\) is a tautology.

- **Proof:** (we must show \((p \land q) \rightarrow p \iff T\))
  \[(p \land q) \rightarrow p \iff \neg(p \land q) \lor p \text{ Useful}\]
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\]

\[
\iff [\neg p \lor \neg q] \lor p \quad \text{DeMorgan}
\]

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(p \land q) \rightarrow p \iff \neg(p \land q) \lor p \quad \text{Useful}
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\[
\iff [\neg p \lor \neg q] \lor p \quad \text{DeMorgan}
\]

\[
\iff [\neg q \lor \neg p] \lor p \quad \text{Commutative}
\]
Using logical equivalences

• **Equivalences can be used in proofs.** A proposition or its part can be transformed using equivalences and some conclusion can be reached.

**Example:** Show that \((p \land q) \to p\) is a tautology.

- Proof: (we must show \((p \land q) \to p \iff T\))
  
  \[
  (p \land q) \to p \iff \neg(p \land q) \lor p
  \]
  
  \[
  \iff \neg p \lor \neg q \lor p \quad \text{DeMorgan}
  \]
  
  \[
  \iff \neg q \lor \neg p \lor p \quad \text{Commutative}
  \]
  
  \[
  \iff \neg q \lor [\neg p \lor p] \quad \text{Associative}
  \]
  
  \[
  \iff \neg q \lor T \quad \text{Useful}
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Using logical equivalences

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\[
(p \land q) \rightarrow p \iff \neg(p \land q) \lor p
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Useful

\[
\iff [\neg p \lor \neg q] \lor p
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DeMorgan

\[
\iff [\neg q \lor \neg p] \lor p
\]

Commutative

\[
\iff \neg q \lor [\neg p \lor p]
\]

Associative

\[
\iff \neg q \lor [T]
\]

Useful

\[
\iff T
\]

Domination

Using logical equivalences

• **Equivalences can be used in proofs.** A proposition or its part can be transformed using equivalences and some conclusion can be reached.

**Example:** Show \((p \land q) \rightarrow p\) is a tautology.

• Alternative proof:

<table>
<thead>
<tr>
<th>p</th>
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<th>p \land q</th>
<th>(p \land q) \rightarrow p</th>
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<tr>
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Using logical equivalences

• **Equivalences can be used in proofs.** A proposition or its part can be transformed using equivalences and some conclusion can be reached.

• Example 2: Show \((p \rightarrow q) \iff (\neg q \rightarrow \neg p)\)

  Proof:
  
  • \((p \rightarrow q) \iff (\neg q \rightarrow \neg p)\)
  
  • \(\iff \ ?\)
Using logical equivalences

• Equivalences can be used in proofs. A proposition or its part can be transformed using equivalences and some conclusion can be reached.

• Example 2: Show \((p \to q) \iff (\neg q \to \neg p)\)
  
  Proof:
  
  • \((p \to q) \iff (\neg q \to \neg p)\)
  • \(\iff \neg(\neg q) \lor (\neg p)\) Useful
  • \(\iff q \lor (\neg p)\) Double negation
  • \(\iff ?\)
Using logical equivalences

- Equivalences can be used in proofs. A proposition or its part can be transformed using equivalences and some conclusion can be reached.

- **Example 2:** Show \((p \rightarrow q) \iff (\neg q \rightarrow \neg p)\)
  
  **Proof:**
  
  - \((p \rightarrow q) \iff (\neg q \rightarrow \neg p)\)
  - \(\iff \neg(\neg q) \lor (\neg p)\) **Useful**
  - \(\iff q \lor (\neg p)\) **Double negation**
  - \(\iff \neg p \lor q\) **Commutative**
  - \(\iff p \rightarrow q\) **Useful**
  
  **End of proof**