CS 441 Discrete Mathematics for CS
Lecture 26

Counting

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Course administration

- **Homework 9 :**
  - **Due on Friday, March 31, 2006**
- **Midterm exam 2**
  - **Tentative: Friday March 31, 2006**
  - **Covers only the material after midterm 1**
    - **Integers (Primes, Division, Congruencies)**
    - **Sequences and Summations**
    - **Inductive proofs and Recursion**
    - **Counting**

Course web page:

http://www.cs.pitt.edu/~milos/courses/cs441/

Permutations

A permutation of a set of distinct objects is an ordered arrangement of the objects. Since the objects are distinct, they cannot be selected more than once. Furthermore, the order of the arrangement matters.

Example:

- Assume we have a set S with n elements. S={a,b,c}.
- **Permutations of S:**
- **a b c   a c b   b a c   b c a   c a b   c b a**

- **k-permutations of S:**
- **ab ac ba bc ca cb**

Number of permutations

- Assume we have a set S with n elements. S={a_1 a_2 … a_n}.
- **Question:** How many different permutations are there?

P(n,n) = n.(n-1)(n-2)…1 = n!

P(n,k)=n.(n-1)(n-2)…(n-k+1) = n!/(n-k)!

Combinations

A *k*-combination of elements of a set is an **unordered** selection of k elements from the set. Thus, a k-combination is simply a subset of the set with k elements.

Example:

- 2-combinations of the set {a,b,c}

a b   a c   b c

↓

a b   covers 2-permutations: **a b** and **b a**

Combinations

Theorem: The number of k-combinations of a set with n distinct elements, where n is a positive integer and k is an integer with $0 \leq k \leq n$ is

$$C(n,k) = \frac{n!}{(n-k)!\,k!}$$

Combinations

Example:

- We need to create a team of 5 player for the competition out of 10 team members. How many different teams is it possible to create?

Answer:

- When creating a team we do not care about the order in which players were picked. It is is important that the player is in. Because of that we need to consider unordered sets of combinations.
- C(10,5) = 10!/(10-5)!5! = (10.9.8.7.6) / (5 4 3 2 1)
  = 2.3.2.7.3= 6.14.3= 6.42= **252**

Binomial coefficients

- The number of k-combinations out of n elements C(n,k) is often denoted as:

$$\binom{n}{k}$$

and reads n choose k. The number is also called **a binomial coefficient.**

- Binomial coefficients occur as coefficients in the expansion of powers of binomial expressions such as

$$(a+b)^n$$

- **Definition:** a binomial expression is the sum of two terms *(a+b).*

Binomial coefficients

Example:

- Expansion of the binomial expression $(a+b)^3$.

$$(a+b)^3 =$$
$$(a+b)(a+b)(a+b) =$$
$$(a^2+2ab+b^2)(a+b) =$$
$$a^3+2a^2b+ab^2+a^2b+2ab^2+b^3 =$$
$$1a^3+3a^2b+3ab^2+1b^3$$

1 3 3 1 ← Binomial coefficients

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

Binomial coefficients

Binomial theorem: Let a and b be variables and n be a nonnegative integer. Then:

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i$$

$$= \binom{n}{0} a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \dots + \binom{n}{n-1} a^1 b^{n-1} + \binom{n}{n} b^n$$

Binomial coefficients

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- **Proof.** The products after the expansion include terms $a^{(n-i)}$ b^i for all i=0,1, …n. To obtain the number of such coefficients note that we have to choose exactly (n-i) a(s) out of the product of n binomial expressions. The number of ways we pull a(s) out of the product is given as: $\binom{n}{n-i} = \binom{n}{i}$
- Thus the theorem holds.

Binomial coefficients

Corrolary: Let n be a nonnegative integer. Then:

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n$$

Binomial coefficients

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Proof:

- Assume a set with n elements:
- C(n,0) = number of subsets of size 0.
- C(n,i) = the number of subsets of size i.
- C(n,n) = the number of subsets of size n.
- The sum of these numbers must give me the number of all subsets of the set n.
- We know it is 2^n so the result follows.

Binomial coefficients

Corrolary:

- Let n be a nonnegative integer. Then:

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} = 0$$

Binomial coefficients

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Proof:

$$\sum_{i=0}^{n} \binom{n}{i} (-1)^i = \sum_{i=0}^{n} \binom{n}{i} (-1)^i 1^{n-i} = ((-1)+1)^n = 0^n = 0$$

Binomial coefficients

Example:

- Show that $\sum_{i=0}^{n} \binom{n}{i} 2^i = 3^n$

- **Answer:**

Binomial coefficients

Example:

- Show that $\sum_{i=0}^{n} \binom{n}{i} 2^i = 3^n$

- **Answer:**

$$\sum_{i=0}^{n} \binom{n}{i} 2^i = \sum_{i=0}^{n} \binom{n}{i} (2)^i 1^{n-i} = (2+1)^n = 3^n$$

Binomial coefficients

Question: We have binomial coefficients for expressions with the power n. Are binomial coefficients for powers of n-1 or n+1 in any way related to coefficients for n ?

- **The answer is yes.**

Theorem:

- Let n and k be two positive integers with . Then it holds:

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

Pascal triangle

Drawing the binomial coefficients for different powers in increasing order gives a Pascal triangle:

	powers
1	
1 1	
1 2 1	**2**
1 3 3 1	**3**

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	5
	6

Pascal triangle

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1 6 15 20 15 6 1	6

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…	7

Permutations with repetitions

- Assume we can pick the object from a set multiple times: that is, an object can be on the position 1, 2, 3, etc

Example:

- 26 letters of alphabet. How many different strings of length k are there?

Answer:

- ?

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Answer:

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Answer:

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Theorem: The number of k-permutations of a set of n objects with repetition is $\mathbf{n^k}$.

Combinations with repetitions

Example:

- Pick four pieces of a fruit from three bowls with apples, pears and oranges. How many possible combinations are possible? List /count all of them?

Answer:

Combinations with repetitions

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Answer:

- **Star and bar approach**
- Apples Pears Oranges
- 3 bowls separated by | |
- Choice 2 apples and 2 pears represented as: ** | ** |
- Choice of 1 apple and 3 oranges: * | | ***

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- Choice of 1 apple and 3 oranges: * | | ***
- **Count:** How many different ways of arranging (3-1)=2 bars and 4 stars are there?
- Total number of positions: 2+4=6

Combinations with repetitions

- **Theorem:** The number of ways to pick n elements from k different groups is:

$$\binom{n-1+k}{n}$$

Combinations with repetitions

- **Theorem:** The number of ways to pick n elements from k different groups is:

$$\binom{n-1+k}{n}$$

- (n+k-1) positions
- n- stars
- **Count:** the number of ways to select the positions of 4 stars.