CS 441 Discrete Mathematics for CS
Lecture 24

Counting

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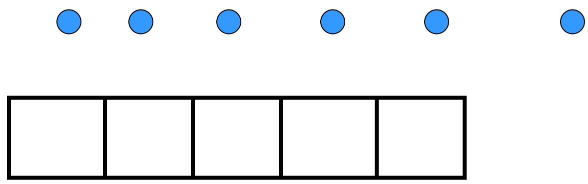
Course administration

- **Homework 8 :**
  - **Due on Friday, March 24, 2006**
- **Midterm exam 2**
  - **Tentative: Friday March 31, 2006**
  - **Covers only the material after midterm 1**
    - **Integers (Primes, Division, Congruencies)**
    - **Sequences and Summations**
    - **Inductive proofs and Recursion**
    - **Counting**

Course web page:
http://www.cs.pitt.edu/~milos/courses/cs441/

Pigeonhole principle

- Assume you have a set of objects and a set of bins used to store objects.
- The **pigeonhole principle** states that if there are more objects than bins then there is at least one bin with more than one object.



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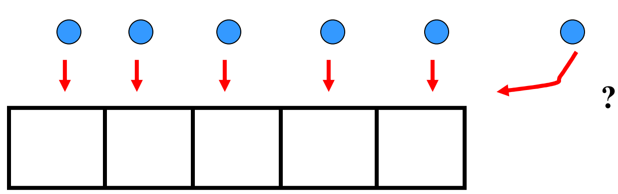

Pigeonhole principle

- **Theorem.** **If there are k+1 objects and k bins. Then there is at least one bin with two or more objects.**

Pigeonhole principle

Example:

- Assume 367 people. Are there (any) two people who has the same birthday?

- How many days are in the year? 365.
- Then there must be at least two people with the same birthday.

Generalized pigeonhole principle

- We can often say more about the number of objects.
- Say we have 5 bins and 12 objects. What is it we can say about the bins and the number of elements they hold?

Generalized pigeonhole principle

Theorem. If N objects are placed into k bins then there is at least one bin containing at least $\lceil N/k \rceil$ objects.

Example. Assume 100 people. Can you tell something about the number of people born in the same month.

Generalized pigeonhole principle

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Example. Assume 100 people. Can you tell something about the number of people born in the same month.

- Yes. There exists a month in which at least $\lceil 100/12 \rceil = \lceil 8.3 \rceil = 9$ people were born.

Generalized pigeonhole principle

Example.

- Show that among any set of 5 integers, there are 2 with the same remainder when divided by 4.

Answer:

- ?

Generalized pigeonhole principle

Example.

- Show that among any set of 5 integers, there are 2 with the same remainder when divided by 4.

Answer:

- Let there be 4 boxes, one for each remainder when divided by 4.
- After 5 integers are sorted into the boxes, there are $\lceil 5/4 \rceil=2$ in one box.

Generalized pigeonhole principle

Example:

- How many students, each of whom comes from one of the 50 states, must be enrolled in a university to guaranteed that there are at least 100 who come from the same state?

Answer:

- ?

Generalized pigeonhole principle

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Answer:

- Let there by 50 boxes, one per state.
- We want to find the minimal N so that $\lceil N/50 \rceil = 100$.
- Letting N=5000 is too much, since the remainder is 0.
- We want a remainder of 1 so that let N=50*99+1=4951.

Permutations

- **A permutation of a set of distinct objects** is an ordered arrangement of the objects. Since the objects are distinct, they cannot be selected more than once. Furthermore, the order of the arrangement matters.

Example:

- Assume we have a set S with n elements. S={a,b,c}.
- **Permutations of S:**
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- Assume we have a set S with n elements. S={a,b,c}.
- **Permutations of S:**
- **a b c   a c b   b a c   b c a   c a b   c b a**

Number of permutations

- Assume we have a set S with n elements. S={a_1 a_2 … a_n}.
- **Question:** How many different permutations are there?
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P(n,n) = n.(n-1)(n-2)…1 = n!

Permutations

Example 1.

- How many permutations of letters {a,b,c} are there?
- Number of permutations is:

  $P(n,n) = P(3,3) = ?$

Permutations

Example 1.

- How many permutations of letters {a,b,c} are there?
- Number of permutations is:

  $P(n,n) = P(3,3) = 3! = 6$
- Verify:

  abc acb bac bca cab cba

Permutations

Example 2

- How many permutations of letters A B C D E F G H contain a substring ABC.

Permutations

Example 2

- How many permutations of letters A B C D E F G H contain a substring ABC.

Idea: consider ABC as one element and D,E,F,G,H as other 5 elements for the total of 6 elements.

Then we need to count the number of permutation of these elements.

$6! = 720$

k-permutations

- ***k*-permutation** is an ordered arrangement of *k* elements of a set.
- The number of *k*-permutations of a set with *n* distinct elements is:

  $P(n,k) = n(n-1)(n-2)\ldots(n-k+1) =$?

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Explanation:

- Assume we have a set S with n elements. S={a_1 a_2 … a_n}.
- The 1st element of the *k*-permutation may be any of the *n* elements in the set.
- The 2nd element of the *k*-permutation may be any of the *n-1* remaining elements of the set.
- And so on. For last element of the *k*-permutation, there are *n-k+1* elements remaining to choose from.

k-permutations

Example:

The 2-permutations of set {*a,b,c*} are:

k-permutations

Example:

The 2-permutations of set $\{a,b,c\}$ are:

ab, ac, ba, bc, ca, cb.

The number of 2-permutations of this 3-element set is

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$P(n,k) = P(3,2) = 3\ (3\text{-}2+1) = 6.$

k-permutations

Example:

Suppose that there are eight runners in a race. The winner receives a gold medal, the second-place finisher receives a silver medal, and the third-place finisher receives a bronze medal. How many different ways are there to award these medals, if all possible outcomes of the race can occur?

Answer: ?

k-permutations

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Suppose that there are eight runners in a race. The winner receives a gold medal, the second-place finisher receives a silver medal, and the third-place finisher receives a bronze medal. How many different ways are there to award these medals, if all possible outcomes of the race can occur?

Answer:

note that the runners are distinct and that the medals are ordered. The solution is $P(8,3) = 8 * 7 * 6 = 8! / (8-3)! = 336$.