CS 441 Discrete Mathematics for CS
Lecture 23

Counting
Pigeonhole principle

Milos Hauskrecht
milos@cs.pitt.edu
5329 Sennott Square

Course administration

- **Homework 7 is due today**
- **Homework 8 is out**
  - **due on Friday, March 24, 2006**

Course web page:

http://www.cs.pitt.edu/~milos/courses/cs441/

Counting

- Assume we have a set of **objects with certain properties**
- **Counting** is used to determine **the number of these objects**

Examples:

- Number of available phone numbers with 7 digits in the local calling area
- Number of possible match starters (football, basketball) given the number of team members and their positions

Basic counting rules

- Counting problems may be very hard, not obvious
- **Solution:**
  - **simplify the solution by decomposing the problem**
- **Two basic decomposition rules:**
  - **Product rule**
    - A count decomposes into a sequence of dependent counts ("each element in the first count is associated with all elements of the second count")
  - **Sum rule**
    - A count decomposes into a set of independent counts ("elements of counts are alternatives")

Product rule

- **Product rule:** If a count of elements can be broken down into a sequence of dependent counts where the first count yields *n1* elements, the second *n2* elements, and kth count *nk* elements, by the product rule the total number of elements is:
  - $n = n1*n2* \ldots * nk$

Example: assume an auditorium with a seat labeled by a letter and numbers in between 1 to 50 (e.g. A23). We want the total number of seats in the auditorium.

- 26 letters and 50 numbers
- Number of seats: 26*50

Sum rule

Sum rule: If a count of elements can be broken down into a set of independent counts where the first count yields *n1* elements, the second *n2* elements, and kth count *nk* elements, by the sum rule the total number of elements is:

- $n = n1+n2+ \ldots + nk$

Example:

You need to travel in between city A and B. You can either fly, take a train, or a bus. There are 12 different flights in between A and B, 5 different trains and 10 buses. How many options do you have to get from A to B?

- We can take only one type of transportation and for each only one option. The number of options:
  - n = 12+5+10

Beyond basic counting rules

- **More complex counting problems** typically require a combination of the sum and product rules.

Example: A login password:

- The minimum password length is 6 and the maximum is 8. The password can consist of either an uppercase letter or a digit. There must be at least one digit in the password.
- How many different passwords are there?

Beyond basic counting rules

Example: A login password.

- The minimum password length is 6 and the maximum is 8. The password can consist of either an uppercase letter or a digit. There must be at least one digit in the password.
- How to compute the number of possible passwords?

Beyond basic counting rules

Step 1:

the total number of valid passwords is by the sum rule:

- **P = P6+P7+P8**
- The number of passwords of length 6,7 and 8 respectively

Step 2

The number of valid passwords of length 6:

P6= P6-digits = P6-all – P6-nodigits

$= 36^6 - 26^6$

Analogically:

P7= P7-digits = P7-all – P7-nodigits

$= 36^7 - 26^7$

P8= P8-digits = P8-all – P8-nodigits

$= 36^8 - 26^8$

Inclusion-Exclusion principle

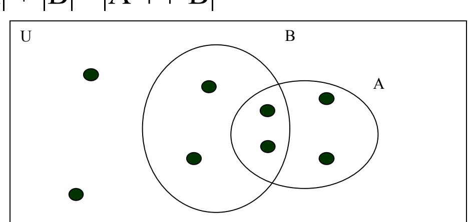
Used in counts where the decomposition yields two count tasks with overlapping elements

- If we used the sum rule some elements would be counted twice

Inclusion-exclusion principle: uses a sum rule and then corrects for the overlapping elements.

We used the principle for the cardinality of the set union.

- $|A \cup B| = |A| + |B| - |A \cap B|$



Inclusion-exclusion principle

Example: How many bitstrings of length 8 start either with a bit 1 or end with 00?

- It is easy to count **strings that start with 1**:
- How many are there?

Inclusion-exclusion principle

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- It is easy to count **strings that start with 1**:
- How many are there? 2^7
- It is easy to count the **strings that end with 00**.
- How many are there?

Inclusion-exclusion principle

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- Is it OK to add the two numbers to get the answer? $2^7 + 2^6$

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- How many of strings were counted twice? 2^5 (1 xxxxx 00)

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- How to deal with it? How to correct for overlap?
- How many of strings were counted twice? 2^5 (1 xxxxx 00)
- Thus we can correct for the overlap simply by using:
- $2^7 + 2^6 - 2^5 = 128 + 64 - 32 =$ **160**

Tree diagrams

Tree: is a structure that consists of a root, branches and leaves.

- Can be useful to represent a counting problem and record the choices we made for alternatives. The count appears on the leaf nodes.

Example:

What is the number of bit strings of length 4 that do not have two consecutive ones.

Tree diagrams

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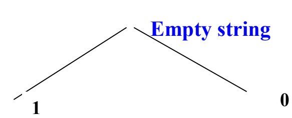

Tree diagrams

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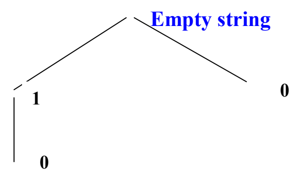

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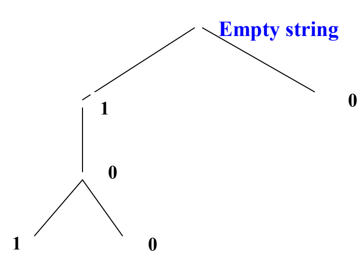

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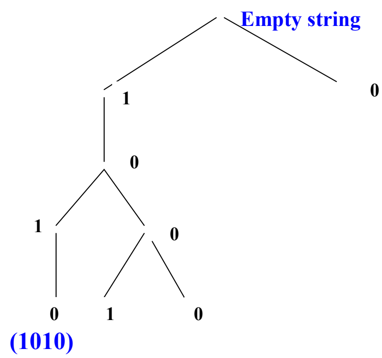

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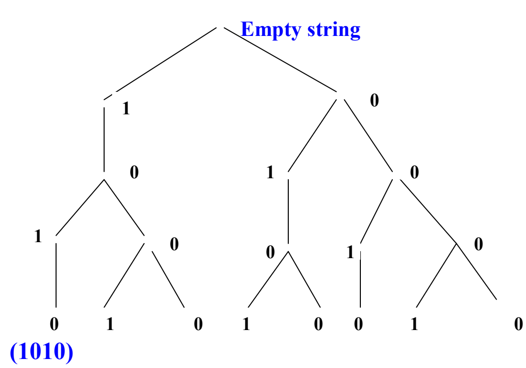

Pigeonhole principle

- Assume you have a set of objects and a set of bins used to store objects. The pigeonhole principle states that if there are more objects than bins then there is at least one bin with more than one object.

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- **Theorem. If there are k+1 objects and k bins. Then there is at least one bin with two or more objects.**

Pigeonhole principle

- Assume you have a set of objects and a set of bins used to store objects. The pigeonhole principle states that if there are more objects than bins then there is at least one bin with more than one object.
- **Theorem.** **If there are k+1 objects and k bins. Then there is at least one bin with two or more objects.**

- **Proof. (by contradiction)**
- Assume that we have k + 1 objects and every bin has at most one element. Then the total number of elements is k which is a contradiction.
- End of proof

Pigeonhole principle

Example:

- Assume 367 people. Are there any two people who has the same birthday?

Pigeonhole principle

Example:

- Assume 367 people. Are there any two people who has the same birthday?
- How many days are in the year? 365.
- Then there must be at least two people with the same birthday.

Generalized pigeonhole principle

- We can often say more about the number of objects.
- Say we have 5 bins and 12 objects. What is it we can say about the bins and number of elements they hold?

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- There must be **a bin with at least 3 elements**.
- Why?

Generalized pigeonhole principle

- We can often say more about the number of objects.
- Say we have 5 bins and 12 objects. What is it we can say about the bins and number of elements they hold?
- There must be **a bin with at least 3 elements**.
- Why?
- There is no bin with more than 3 elements. Max number of elements we can have in 5 bins is 10. We need to place 13 so at least one bin should have at least 3 elements.

Generalized pigeonhole principle

Theorem. If N objects are placed into k bins then there is at least one bin containing at least $\lceil N/k \rceil$ objects.

Example. Assume 100 people. Can you tell something about the number of people born in the same month.

- Yes. There exists a month in which at least $\lceil 100/12 \rceil = \lceil 8.3 \rceil = 9$ people were born.